Iranian Journal of Astronomy and Astrophysics
Vol. 3, No. 2, Autumn 2016
©Available online at http://journals.du.ac.ir
DOI:10.22128/ijaa.2016.82

Iranian Journal of Astronomy and Astrophysics

Spherically Symmetric Solutions in a New Braneworld Massive Gravity Theory

Amir Asaiyan[1] · Kourosh Nozari[2]

[1] Department of Physics, Faculty of Basic Sciences, University of Mazandaran, P.O.Box 47416-95447, Babolsar, Iran ;
email: a.asaiyan@umz.ac.ir (Corresponding Author)

[2] email: knozari@umz.ac.ir

Abstract. In this paper, a combination of the braneworld scenario and covariant de Rham-Gabadadze-Tolley (dRGT) massive Gravity theory is proposed. In this setup, the five-dimensional bulk graviton is considered to be massive. The five dimensional nonlinear ghost-free massive gravity theory affects the 3-brane dynamics and the gravitational potential on the brane. Following the solutions with spherical symmetry on the brane, the full field equations together with the generalized Israel-Darmois junction conditions on the brane and their weak field limits are presented in details. Generally, the theory has four Stückelberg fields along with the components of physical metric. Although analytical solutions of these equations are impossible in general, by considering some simplifying assumptions, two classes of four-dimensional spherically symmetric solutions on the brane with different background Stückelberg fields are obtained.

Keywords: Braneworld Gravity, Massive Gravity, Black Holes

1 Introduction

The accelerating expansion of the Universe has forced us to challenge with our understanding of the fundamental physics [1, 2, 3]. In the last two decades, there has been considerable interest in theories of gravitation that modify the Einstein's gravity at very large distance scales. These theories could explain the present day acceleration, without including a cosmological constant or an exotic matter content. Adding one or even many extra spatial dimensions to the 4D Einstein's theory of gravity may lead to the interesting phenomenological results. The Braneworld model is an extra dimensional theory, in which our universe is a 3-brane embedded in a five-dimensional spacetime called the bulk [4, 5]. All matter fields reside on the brane, but gravitons can travel into the extra dimension. The Dvali-Gabadadze-Porrati (DGP) model [6] is an interesting braneworld model in which the bulk is empty, the extra dimension is infinitely large. Also a 4D Einstein-Hilbert term in the braneworld action exists. The model has attractive results from cosmological viewpoint because gravity on the brane is weakened and becomes five-dimensional at large scales, $r \gg r_c$ (where r_c is the DGP crossover distance), while on small scales, gravity is effectively bounded to the brane and 4D dynamics is regained. It contains a self-accelerating branch of the solutions which can explain late time cosmic speed up [7, 8, 9]. From the 4D perspective, gravity on the brane is mediated by an infinite number of Kaluza-Klein (KK) modes that have not discontinuities. The 4D Einstein-Hilbert term on the brane will suppress the wave functions of heavier KK modes, so that they do not participate in the gravitational interactions on the brane at observable distances [10]. The 4D gravity on the brane is mediated by a massless zero mode, whereas the couplings of the heavy KK modes to ordinary matter

are suppressed.
Due to the cosmological constant problem, we should look for a technically natural way of describing cosmic acceleration. The massive gravity theories are other kinds of modified gravity theories, in which a small graviton mass may lead to an IR modification of gravity with an accelerated expansion without a small cosmological constant. The recent experiments GW150914 and GW151226 [11, 12] by LIGO, were able to detect the gravitational waves and put an upper limit on the graviton mass, i.e. $m < 1.2 \times 10^{-22}$ eV [13]. At the linearized level, the Fierz-Pauli (FP) graviton mass term is the only Lorentz-invariant mass term which after quantization does not generate ghosts in flat space [14]. However, choosing a Fierz-Pauli mass term for the graviton will lead to the well known vDVZ discontinuity [15, 16]. The coupling of the longitudinal polarization of the massive graviton to trace of the energy-momentum tensor in the limit of zero graviton mass is responsible for this discontinuity, such that the tensor structure of the gravitational interaction deviates from that of Einstein gravity. To restore the continuity of Fierz-Pauli massive gravity theory at graviton mass $m = 0$, two different approaches have been proposed. The first one, which was first pointed out by Vainshtein [17, 18], is to consider nonlinear effects. The other way is to consider a curved maximally symmetric spacetime (dS or AdS) with $\frac{m}{H} \to 0$ [19, 20]. In 1972, Vainshtein noted that there is a radius r_V, known as Vainshtein radius, around a massive source, inside of it the linear approximation breaks down and at massless limit r_V goes to infinity [17]. Therefore, the nonlinear terms are important in the limit $m \to 0$. However, Boulware and Deser argued that the non-linear terms cause a scalar field with wrong sign kinetic term, known as Boulware-Deser (BD) ghost [21]. At the classical level, this scalar may not be a problem due to non-linear effects [17, 18], but at the quantum level the theory becomes strongly coupled [22] at energy scale $\Lambda_5 \equiv (m^4 M_P)^{1/5}$. By adding higher order operators, this scale can be raised to order $\Lambda_3 \equiv (m^2 M_P)^{1/3}$.

In 2010, de Rham and Gabadadze studied generic extensions of the Fierz-Pauli Lagrangian by higher-order interactions of the massive spin-2 fluctuation $h_{\mu\nu}$ [23]. Their analysis went to quintic order in the longitudinal component of $h_{\mu\nu}$ and demonstrated that its interactions could in fact be made ghost-free in a decoupling limit. The decoupling limit analysis relies heavily on the aforementioned Goldstone boson analogously suggested by Arkani-Hamed, Georgi and Schwartz [22]. de Rham, Gabadadze and Tolley (dRGT) [24] completed their investigations by presenting a nonlinear theory of massive gravity whose decoupling limit is ghost-free for all nonlinear self-interactions of the longitudinal component [24, 25, 26, 27]. The dRGT theory is the unique ghost-free theory for massive graviton and new kinetic interactions are not consistent [28, 29]. See [30, 31, 32] for recent reviews on all aspects of massive gravity and bimetric theories. In the context of the dRGT nonlinear covariant massive gravity model [23, 24], some self-accelerating solutions have been discovered [33, 34, 35, 36, 37, 38]. Dynamics of the scalar mode of a massive graviton in four-dimensions has been studied in detail in [36], showing that a non-trivial configuration for this field leads to self-acceleration. Scalar fluctuations around these self-accelerating configurations are proved to be free of ghosts.

It is worthwhile to note that one way in which a massive graviton naturally arises is higher dimensional scenarios. A theory of gravity with compactified extra dimensions can be viewed as a four dimensional theory of multiple gravitons, i.e. KK modes. An alternative to the KK paradigm was the ADD model [39, 40] in which one (or more) extra dimension could emerge from a theory of a finite number of massive gauge fields or gravitons living in four dimensions. Their idea, named "Dimensional Deconstruction", can be viewed as taking a five dimensional gauge or gravity theory and discretizing the extra dimension(s).

It has been shown that Dimensional Deconstruction is equivalent to a truncation of the KK tower at the nonlinear level [28]. It has been shown that the DGP model is closely related to massive gravity. In this model, the 4D graviton propagator on the brane in the Gaussian normal coordinates is similar to the propagator for 4D massive gravity with graviton mass $m^2 = (\frac{1}{r_c})\sqrt{-\Box}$, where $r_c \equiv (M_p^2/2M_5^3)$ is the DGP crossover length scale and $\Box$ is the four-dimensional d'Alembertian. In other words, the graviton acquires a soft mass, or resonance effectively, in the DGP model. The induced gravity term in the brane action acts as a kinetic term for a 4D graviton while the bulk Einstein-Hilbert term acts as a gauge invariant mass term. Therefore, the vDVZ discontinuity problem is also present in the DGP model. Here, the massless limit converts to the limit $r_c \to \infty$. As argued by Vainshtein, at distances smaller than the radius r_V, the linearization breaks down and by considering non-linear effects, we can restore the predictions of GR on the brane [17, 18, 41, 42]. However, the DGP model has some consistency problems. The normal branch of the DGP theory is free of ghosts and instabilities, but the self-accelerating branch is completely unstable [43, 44, 45]. The DGP model has strong interactions at energy scale $\Lambda \sim (M_p/r_c^2)^{1/3}$. From the 4D point of view, there is an extra scalar degree of freedom π that contributes to the extrinsic curvature of the brane as $K_{\mu\nu} \propto \partial_\mu\partial_\nu\pi$ [43, 44]. Indeed, this scalar is a brane bending mode that interacts strongly at momenta of order Λ. In the decoupling limit of the DGP model, in which Λ is kept fixed, only the π sector exists and all other degrees of freedom decouple. This limit reduces to the cubic Galileon for the helicity-0 mode π [46].

The works done by Gabadadze and de Rham before proposing the interesting dRGT theory have shown that the introduction of the spurious extra dimension provides a geometrical interpretation of massive gravity, for which non-linearities can be tracked down explicitly [47, 48]. By studying massive gravity from extra dimensional point of view, we can better understand certain aspects of the dRGT theory [23, 24] and its bigravity [49] and multi-gravity [50] extensions. In 2009, Gabadadze considered an extension of GR by an auxiliary non-dynamical extra dimension and showed that the obtained gravitational equations could have a self-accelerated solution, which is due to a new mass parameter m. The auxiliary dimension gives an extrinsic curvature to the 4D space-time and the extrinsic curvature is responsible for creating the mass term. The special structure $[K]^2 - [K^2]$ arose from the Gauss equation for the bulk Ricci scalar ensures the Fierz-Pauli structure which is ghost-free at the linearized level [47]. de Rham and Gabadadze [48, 51] verified that the theory in the decoupling limit is free of the Boulware-Deser ghost to cubic order. In ref [28], it was shown that the ghost-free models of massive gravity and their multi-graviton extensions can follow from considering higher dimensional extension of GR in the Einstein-Cartan form on a discrete extra dimension. Indeed, discretizing the extra dimension in the vielbein language can automatically generate the square root structure characteristic of the dRGT model, i.e. $\mathcal{K}^\nu_\mu$, [28]. Indeed, the expression for the discretized extrinsic curvature coincides with $\mathcal{K}^\nu_\mu$.

By considering the above arguments, now giving a mass to the graviton in Higher-dimensional theories and exploring the overall effects of massive gravity and extra dimension could be interesting from theoretical and phenomenological viewpoints. The final results may have some relations with the multi-metric theories and then lead to physically interesting predictions. In 2004, Chacko et al., considered a braneworld setup in warped anti-de Sitter spacetime (Randall-Sundrum (RS) two-brane model [52]) with a mass term for the graviton on the infrared brane [53]. The predictions of this theory coincide with the results of GR at distances smaller than the infrared scale but at longer distances a theory of massive gravity exists. However, in the low energy limit of the theory, there is a ghost, which corresponds to the radion field. In Ref. [54], both of the bulk and the brane mass terms were introduced in the action of the RS two-brane model to quadratic order to modify the profile of the graviton zero-mode in the extra dimensions. It was found that for a particular choice of parameters, there is an IR-peaked zero-mode, i.e. the graviton can be localized on the IR brane. In 2014, a braneworld scenario has been investigated in which the infinite-volume bulk graviton was massive [55]. The bulk graviton can be as heavy as the bulk Planck scale which is much larger than the inverse Hubble size. The 4D induced gravity term on the brane shields the brane matter from both strong bulk gravity and large bulk graviton mass. Higher-dimensional gravity at large distances are not obtained on the brane in this setup and at distances above the bulk Planck length scale, the 4D graviton on the brane acquires a small mass. The author of [55] considered a mass potential that arose via the gravitational Higgs mechanism, such that a general quadratic potential in terms of perturbation tensor h_{AB} was introduced in the bulk action. In this extension of the DGP model, even for the case of ghost-free Fierz-Pauli bulk mass term, the 4D tensor structure on a 3-brane could be obtained [55]. Here, the key point is that the trace $h \equiv \eta^{AB} h_{AB}$ is perturbatively a ghost. However, it was shown that the non-perturbative Hamiltonian is bounded from below and there is no ghost in full nonlinear theory [56, 57, 58].

With these detailed preliminaries which are necessary for a reader to understand forthcoming arguments in this paper, we consider a combination of the DGP braneworld and dRGT massive gravity models, by introducing a five dimensional nonlinear ghost-free potential in the bulk action. In this setup, our universe is a 3-brane embedded in a 5D bulk where the extra spatial dimension is large. A 5D ghost-free massive gravity theory propagates nine degrees of freedom (DOF) and the extra four DOFs added to the five DOFs of 5D massless graviton, which is effectively equivalent to a 4D softly massive graviton, are the extra polarizations of the 5D massive graviton. We considered the induced gravity term on the brane action, because this term in the DGP setup acts as a kinetic term for the 4D graviton. The 5D extension of dRGT theory is free of ghosts and we want to explore the effects of this nonlinear theory on the brane dynamics and the effective 4D gravitational interactions on the brane. For this purpose, the full 5D field equations and their weak field limits have been studied. Our focus is on the solutions with spherical symmetry on the brane. The full nonlinear equations of motion in the presence of the unknown stückelberg fields are generally very complicated to solve for analytical solutions, unless we consider some simplifying assumptions. So, to have some intuition and to be more clarified, we have adopted step by step some reasonable and simplifying assumptions to find a class of four-dimensional spherically symmetric solutions on the brane. We considered two simplified linear theories in both unitary and non-unitary gauges and found in both cases a flat solution on the brane with different background Stückelberg fields. In non-unitary gauge we restricted ourself to special choices of the free parameters of the theory. We note that general massive braneworld solutions, resulting from the full nonlinear theory, should reduce to the massless braneworld solution in the limit of zero bulk graviton mass as has been studied in [59]. We are attempting to follow new approaches, such as solving the nonlinear field equations numerically or finding the effective 4D field equations on the brane [60, 61], to examine the Vainshtein mechanism in our model.

2 Braneworld Massive Gravity

In braneworld scenarios, we assume that our (1+3)-dimensional spacetime is a domain wall embedded in a five-dimensional spacetime called the bulk [4, 5]. All matter fields live on the brane but only gravitons can travel into the bulk. In the DGP braneworld model, the bulk is empty, the extra dimension is infinitely large and a 4D Einstein-Hilbert term exists on the brane action [6]. In our braneworld massive gravity model, we introduce a mass potential to the bulk action, which is a 5D extension of the dRGT's 4D nonlinear ghost-free massive gravity theory [23, 24]. We consider a 3-brane Σ embedded in the five-dimensional massive bulk $\mathcal{M}$. The total action is

$$S = \frac{M_5^3}{2} \int_{\mathcal{M}} d^5X \sqrt{-g} \left({}^{(5)}R + m_g^2 \,\mathcal{U}(g, \mathcal{K}) \right) + S_{brane}, \tag{1}$$

where S_{brane} is the 3-brane action defined as

$$S_{brane} = \frac{M_p^2}{2} \int_{\Sigma} d^4x \sqrt{-q}\; {}^{(4)}R + \int_{\Sigma} d^4x \sqrt{-q}\, \mathcal{L}_4^{matt} + \int_{\Sigma} d^4x \sqrt{-q}\, \frac{K}{\kappa_5^2}\,. \tag{2}$$

g_{AB} is the 5D bulk metric with corresponding Ricci tensor given by ${}^{(5)}R_{AB}$. X^A, $A = 0, 1, 2, 3, 5$ are the coordinates in the bulk. The brane has induced metric $q_{\mu\nu}$ with corresponding Ricci tensor ${}^{(4)}R_{\mu\nu}$. $\mathcal{L}_4^{matt}$ is the matter Lagrangian localized on the brane. We note also that the bulk Planck mass M_5 and the 4-dimensional Planck scale M_p are defined as $\kappa_5^2 = 8\pi G_{(5)} = M_5^{-3}$ and $\kappa_4^2 = 8\pi G_{(4)} = M_p^{-2}$. $\mathcal{U}$ is a dimensionless "potential" for the metric g_{AB} that makes bulk graviton massive, where the dimension-full parameter m_g sets the graviton mass scale. This potential depends on three dimensionless arbitrary parameters α_3, α_4 and α_5 and is composed of four parts,

$$\mathcal{U}(g, \mathcal{K}) = \sum_{n=2}^{5} \alpha_n \mathcal{U}_n(\mathcal{K}) = \mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4 + \alpha_5 \mathcal{U}_5, \tag{3}$$

where $\alpha_2 = 1$. The tensor $\mathcal{K}_A^B$ is

$$\mathcal{K}_A^B = \delta_A^B - \sqrt{g^{BC}(g_{CA} - H_{CA})} = \delta_A^B - \sqrt{g^{BC} f_{ab} \partial_C \phi^a \partial_A \phi^b}. \tag{4}$$

The potential (3) is unique and no further polynomial terms can be added to the action without introducing the BD ghost [23, 24, 25, 26, 27]. The sum is finite and stops at $n = 5$, since the total derivative combinations vanish for $n > D = 5$ [24, 31]. It was shown that this is the most general potential for a ghost-free theory of massive gravity [62]. f_{ab} is the fiducial (or reference) metric, which we assume to be the Minkowski metric, $\eta_{\mu\nu}$, and ϕ^a are the Stückelberg scalar fields introduced to give a manifestly diffeomorphism invariant description [22]. Under a diffeomorphism $\delta X^A = \xi^A(X)$, the Stückelberg fields ϕ^0, ϕ^i ($i = 1, 2, 3, 5$) transform as simple scalars. The tensor h_{AB} represents the fluctuations of bulk metric about Minkowski reference metric, $h_{AB} = g_{AB} - \eta_{AB}$, and H_{AB} corresponds to the covariantization of metric perturbations, defined as $H_{AB} = g_{AB} - \partial_A \phi^a \partial_B \phi^b \eta_{ab}$. The square root is formally understood as $\sqrt{W}^A_C \sqrt{W}^C_B = W^A_B$. The four polynomial terms $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{U}_4$, and $\mathcal{U}_5$ depend on the metric g and Stückelberg fields ϕ^a as

$$\mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \tag{5}$$

$$\mathcal{U}_3 = \frac{1}{3}[\mathcal{K}]^3 - [\mathcal{K}^2][\mathcal{K}] + \frac{2}{3}[\mathcal{K}^3], \tag{6}$$

$$\mathcal{U}_4 = \frac{1}{12}[\mathcal{K}]^4 - \frac{1}{2}[\mathcal{K}^2][\mathcal{K}]^2 + \frac{2}{3}[\mathcal{K}^3][\mathcal{K}] + \frac{1}{4}[\mathcal{K}^2]^2 - \frac{1}{2}[\mathcal{K}^4], \tag{7}$$

$$\mathcal{U}_5 = \frac{1}{60}[\mathcal{K}]^5 - \frac{1}{3}[\mathcal{K}^3][\mathcal{K}^2] + \frac{1}{3}[\mathcal{K}^3][\mathcal{K}]^2 - \frac{1}{6}[\mathcal{K}^2][\mathcal{K}]^3 - \frac{1}{2}[\mathcal{K}][\mathcal{K}^4] + \frac{1}{4}[\mathcal{K}][\mathcal{K}^2]^2 + \frac{2}{5}[\mathcal{K}^5], \quad (8)$$

where the square brackets are defined as

$$[\mathcal{K}] \equiv \mathrm{tr}\mathcal{K}^B_A, \quad [\mathcal{K}]^2 \equiv \left(\mathrm{tr}\mathcal{K}^B_A\right)^2, \quad [\mathcal{K}^2] \equiv \mathrm{tr}\mathcal{K}^B_C\mathcal{K}^C_A. \quad (9)$$

We chose a coordinate y for the extra dimension so that our 3-brane is localized at $y = 0$. Variation of the action (1) with respect to the bulk metric leads to the modified 5D field equations in the bulk as [56, 57, 58]

$$^{(5)}G_{AB} + m_g^2\,\overline{X}_{AB} = \kappa_5^2\ {}^{(loc)}T_{AB}\delta(y), \quad (10)$$

where $\overline{X}_{AB}$ is the effective energy-momentum tensor due to the graviton mass and expressed as

$$\overline{X}_{AB} = X_{AB} + \sigma Y_{AB}, \quad (11)$$

with

$$X_{AB} = -\frac{1}{2}(\alpha\mathcal{U}_2 + \beta\mathcal{U}_3)g_{AB} + \tilde{X}_{AB}, \quad (12)$$

$$\tilde{X}_{AB} = \mathcal{K}_{AB} - [\mathcal{K}]g_{AB} - \alpha\Big\{\mathcal{K}^2_{AB} - [\mathcal{K}]\mathcal{K}_{AB}\Big\} + \beta\Big\{\mathcal{K}^3_{AB} - [\mathcal{K}]\mathcal{K}^2_{AB} + \frac{\mathcal{U}_2}{2}\mathcal{K}_{AB}\Big\}, \quad (13)$$

$$Y_{AB} = -\frac{\mathcal{U}_4}{2}g_{AB} + \tilde{Y}_{AB}, \quad (14)$$

$$\tilde{Y}_{AB} = \frac{\mathcal{U}_3}{2}\mathcal{K}_{AB} - \frac{\mathcal{U}_2}{2}\mathcal{K}^2_{AB} + [\mathcal{K}]\mathcal{K}^3_{AB} - \mathcal{K}^4_{AB}. \quad (15)$$

The new parameters α, β, and σ are defined as $\alpha = 1 + \alpha_3$, $\beta = \alpha_3 + \alpha_4$, $\sigma = \alpha_4 + \alpha_5$, and the indices are raised and lowered by the "physical" metric g_{AB}, so that $\mathcal{K}_{AB} = g_{AC}\mathcal{K}^C_B$, $\mathcal{K}^2_{AB} = g_{AD}\mathcal{K}^D_C\mathcal{K}^C_B$, etc.

The effective localized energy-momentum tensor on the brane including the contribution from the induced 4D Einstein-Hilbert term on the brane is

$$^{(loc)}T_{AB} = g^\mu_A g^\nu_B(-\frac{1}{\kappa_4^2})\sqrt{\frac{-q}{-g}}\left(^{(4)}G_{\mu\nu} - \kappa_4^2\ {}^{(4)}T_{\mu\nu}\right). \quad (16)$$

where $^{(5)}G_{AB}$ and $^{(4)}G_{AB}$ denote the Einstein tensors constructed from the bulk and the brane metrics respectively. The tensor $q_{AB} = g_{AB} - n_A n_B$ is the induced metric on the brane Σ with n_A the normal vector on this hypersurface. The field equations in the bulk $(y \neq 0)$ take the following form

$$^{(5)}G_{AB} = {}^{(5)}R_{AB} - \frac{1}{2}\,{}^{(5)}R\,g_{AB} = -m_g^2\,\tilde{X}_{AB}. \quad (17)$$

Moreover, if the components of $\tilde{X}_{AB}$ be continuous across $y = 0$, the following modified (due to the presence of induced gravity on the brane) Israel-Darmois junction conditions, as a boundary condition for the field equations in the bulk, would be obtained

$$[K^\nu_\mu] - \delta^\nu_\mu[K] = -\kappa_5^2\ {}^{(loc)}T^\nu_\mu = \left(\frac{\kappa_5^2}{\kappa_4^2}\right){}^{(4)}G^\nu_\mu - \kappa_5^2\ {}^{(4)}T^\nu_\mu, \quad (18)$$

where $K_{\mu\nu} = \frac{1}{2}\partial_y(g_{\mu\nu})$ is the extrinsic curvature of the brane and brackets denote jump across the brane ($y = 0$). We assume a $\mathbf{Z}_2$-symmetry on reflection around the brane, thus the Israel-Darmois junction conditions become

$$\overline{K}_{\mu}^{\ \nu} - \overline{K}\delta_{\mu}^{\ \nu} = r_c\, {}^{(4)}G_{\mu}^{\nu} - \frac{\kappa_5^2}{2}\, {}^{(4)}T_{\mu}^{\nu}, \tag{19}$$

where $r_c = \frac{\kappa_5^2}{2\kappa_4^2} = \frac{M_p^2}{2M_5^3}$ is the well-known DGP crossover distance, and by definition $\overline{K}_{\mu}^{\ \nu} = K_{\mu}^{\ \nu}(y = 0^+) = -K_{\mu}^{\ \nu}(y = 0^-)$.

After presentation of general field equations in the proposed setup, now we seek for some spherically symmetric solutions on the brane.

3 Spherically Symmetric Solutions

Here, we consider the static spherically symmetric configurations on the brane and our concentration is on the issue of braneworld black holes, i.e. finding the bulk and the brane metric when a spherically symmetric energy-momentum distribution is localized on the brane. In our previous work [59], black hole solutions in warped DGP braneworld model with a cosmological constant term in the bulk were obtained (see [63, 64, 65, 66] for further black hole solutions in braneworld scenarios). We found a 5D black string solution for the bulk metric, which reduces to 4D Schwarzschild-AdS solution on the brane. The 4D AdS curvature radius is proportional to r_c, therefore the Schwarzschild solution is recovered on the brane in the limit $r_c \to \infty$ [59]. As we already noted, the DGP model is closely related to massive gravity and the 4D graviton propagator on the brane is similar to the propagator for 4D massive graviton. In the dRGT theory with a Minkowski reference metric, a class of non-bidiagonal Schwarzschild-dS solutions was found in [33, 34]. In this theory, for a special choice of free parameters of the action, the Schwarzschild-dS type of black hole solutions was obtained in ref [67, 35], where the mass term behaves similar to the cosmological constant term in GR. For this choice of parameters, the Bianchi identity is automatically satisfied for a certain diagonal and time-independent metrics in spherical polar coordinates, whereas the kinetic terms for both the vector and scalar fluctuations vanish in the decoupling limit. Although it was shown that the linearized solutions of GR can be reproduced below the Vainshtein radius in a certain region of parameter space, the metric here is accompanied by nontrivial backgrounds for the Stückelberg fields. The vector and scalar modes A^μ and π of massive gravitons are the nonunitary parts of the background Stückelberg fields [35], i.e. $x^\mu - \phi^\mu = (m\, A^\mu + \partial^\mu \pi)/\Lambda^3$. For reviewing the black hole solutions in massive gravity see refs. [68, 69, 70, 71].

All of these papers have focused only on the four-dimensional dRGT theory [23, 24], in which only the usual graviton terms, $\mathcal{U}_i$ ($i = 2 - 4$), are considered. For spherically symmetric solutions in extra dimensional setups, some types of black hole solutions for dRGT massive gravity with their thermodynamical properties have been investigated in d-dimensional spacetimes ($d \geq 3$) in refs. [72, 73, 74, 75, 76, 62]. The behavior of massive graviton terms for some cosmological solutions such as the FLRW, Bianchi type I, and also Schwarzschild-Tangherlini-(A)dS metrics in a specific five-dimensional nonlinear massive gravity and bigravity models have been clarified in Refs. [62, 77]. In ref. [78], it was argued that giving a space-dependent mass to the 5D graviton, which depends on the extra-dimensional coordinate, can localize Einstein gravity on a 3-brane embedded in a 5D

Minkowski space. They focused on the quadratic Fierz-Pauli Lagrangian for 5D metric perturbations and explored the linearized equations of motion for 4D scalar, vector and tensor modes. They showed that there is no ghost on the brane and conserved matter on the brane does not couple to the scalar massless mode. The nonlinear extension of the theory has not been studied yet.

We want to find a 4D spherically symmetric solution for our nonlinear massive braneworld setup and separately determine the effects of bulk graviton mass term and also the large extra dimension on the gravitational interactions on the brane. We expect that the predictions of GR and DGP model be reproduced in appropriate limits, i.e. $m \to 0$ limit for recovering the DGP results and $r_c \to \infty$ limit in addition to the previous one for recovering GR on the brane. The issues of the vDVZ discontinuity and the Vainshtein mechanism to resolve it should be carefully studied. The effects of bulk nonlinear terms and the brane bending modes play important roles in these limits. To obtain black hole solutions in a braneworld scenario, generally there are two different approaches. In the first approach, as we explained in last section, dynamics and geometry of the whole bulk spacetime are primarily considered; then the dynamics on the brane is extracted by using the Israel-Darmois matching conditions. The second approach is to obtain the effective four-dimensional field equations on the brane firstly and then try to extend these solutions into the bulk [60, 61]. Here, we will follow the first approach. Therefore, to choose an appropriate 5D line element which is spherically symmetric on the brane, we review the 4D black hole solutions of the original dRGT theory. In this case, the ansatz for the static spherically symmetric solutions is the same as in GR. The only subtlety consists in getting the correct configuration for the four scalar fields. Regarding the vacuum solution of the theory, ($\phi^a = x^\mu\, \delta^a_\mu$ and $g_{\mu\nu} = \eta_{\mu\nu}$), the spherically symmetric line element and the four scalar fields for 4D massive gravity models can be written as follows

$$ds^2 = -\alpha(r)dt^2 + 2\delta(r)dtdr + \beta(r)dr^2 + \chi(r)\Big(d\theta^2 + \sin^2(\theta)d\varphi^2\Big), \tag{20}$$

$$\phi^0 = t + h(r), \quad \phi^i = \phi(r)\frac{x^i}{r}. \tag{21}$$

In the unitary gauge, the scalar fields are $\phi^a = x^a = (t, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. Therefore, in this gauge $h(r) = 0$ and $\phi(r) = r$. The field configuration is invariant under two residual coordinate transformations. The first one is an arbitrary change of the radial coordinate $r \to \widetilde{r} = \widetilde{r}(r)$, which allows to set either $\chi(r) = r^2$ or $\phi(r) = r$. The second one is the redefinition of the time variable $t \to \widetilde{t} = t + \tau(r)$, which allows to cancel either $\delta(r)$ or $h(r)$. In our five dimensional braneworld theory, we can choose a coordinate system in which the brane is located at $y = 0$ and the 5D metric with spherical symmetry on the brane are as follows

$$ds_5{}^2 = -e^{\nu(r,y)}dt^2 + e^{\lambda(r,y)}dr^2 + r^2e^{\mu(r,y)}d\Omega^2 + dy^2\,, \tag{22}$$

where the 5D Stückelberg fields are

$$\phi^0 = t,\ \phi^i = \phi(r)\frac{x^i}{r},\ \phi^5 = y. \tag{23}$$

As compared to ordinary Braneworld theories, this configuration contains an additional radial function $\phi(r)$, which should be determined. The matter content of the 3-brane universe is considered to be a localized spherically symmetric untilted perfect fluid (e.g. a star) with

$$^{(4)}T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pq_{\mu\nu}\,, \tag{24}$$

where u^μ stands for the 4-velocity of the fluid and $\rho = p = 0$ for $r > R$. Nevertheless, since we want to obtain static black hole solutions outside the star (that is, for $r > R$), in these regions the brane is empty. With the ansatz (22) and (23), the components of $\mathcal{K}^B_A$ would take the following form

$$\mathcal{K}^B_A = \mathrm{diag}\Big(1-(e^{-\frac{\nu}{2}}), 1-(\phi' e^{-\frac{\lambda}{2}}), 1-(\frac{\phi}{r}e^{-\frac{\lambda}{2}}), 1-(\frac{\phi}{r}e^{-\frac{\lambda}{2}}), 0\Big). \tag{25}$$

By using these components, we can obtain the total derivative combinations $\mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$ and $\mathcal{U}_5$. We have found that the term $\mathcal{U}_5$ vanishes for this configuration. Consequently, the components of $\overline{X}_{AB}$ can be obtained analytically although their expressions are so lengthy. The Einstein tensor components are nonlinear and second order in terms of ν, λ, μ and their partial derivatives. To find some analytical solutions, firstly we consider the *weak-field regime* (i.e. far enough from the source localized on the brane). In this respect, we will find solutions in the regimes where $|\nu|$, $|\lambda|$ and $|\mu|$ are small quantities compared to unity; that is, $|\nu|, |\lambda|, |\mu| \ll 1$. By adopting this assumption, we linearize our field equations with respect to these functions. Now, by putting the metric (22) into the bulk field equations (17) and keeping only the leading-order terms, we obtain the (tt), (rr), $(\theta\theta)$, (yy), and (ry) components of the bulk field equations respectively as follows:

$$\begin{aligned}
&2(\mu-\lambda)+2r^2\mu_{rr}+6r\mu_r-2r\lambda_r+r^2(\lambda_{yy}+2\mu_{yy})\\
&+2m_g^2r^2\Bigg\{\Big[3+3\alpha+\beta-(1+2\alpha+\beta)(\phi'+2\frac{\phi}{r})+(\alpha+\beta)(\frac{\phi^2}{r^2}+2\frac{\phi\phi'}{r})-\beta\frac{\phi'\phi^2}{r^2}\Big](1+\nu)\\
&+\Big[(1+2\alpha+\beta)\frac{\phi'}{2}-(\alpha+\beta)\frac{\phi\phi'}{r}+\beta\frac{\phi'\phi^2}{2r^2}\Big]\lambda+\Big[(1+2\alpha+\beta)\frac{\phi}{r}-(\alpha+\beta)(\frac{\phi\phi'}{r}+\frac{\phi^2}{r^2})+\beta\frac{\phi'\phi^2}{r^2}\Big]\mu\Bigg\}=0\,,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&2(\lambda-\mu)-2r\mu_r-2r\nu_r-r^2(\nu_{yy}+2\mu_{yy})+2m_g^2r^2\Bigg\{\Big[-(\alpha+2)+2(\alpha+1)\frac{\phi}{r}-\alpha\frac{\phi^2}{r^2}\Big](1+\lambda)\\
&+\Big[-\frac{1}{2}(1+2\alpha+\beta)+(\alpha+\beta)\frac{\phi}{r}-\frac{1}{2}\beta\frac{\phi^2}{r^2}\Big]\nu+\Big[-(\alpha+1)\frac{\phi}{r}+\alpha\frac{\phi^2}{r^2}\Big]\mu\Bigg\}=0\,,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&-r^2(\nu_{rr}+\mu_{rr})-r\nu_r-2r\mu_r+r\lambda_r-r(\nu_{yy}+\lambda_{yy}+\mu_{yy})+2m_g^2r^2\Bigg\{-(\alpha+2)+(\alpha+1)(\frac{\phi}{r}+\phi')-\alpha\frac{\phi\phi'}{r}\\
&+\Big[-\frac{1}{2}(1+2\alpha+\beta)+\frac{1}{2}(\alpha+\beta)(\frac{\phi}{r}+\phi')-\frac{1}{2}\beta\frac{\phi\phi'}{r}\Big]\nu+\Big[-\frac{1}{2}(1+\alpha)\phi'+\frac{1}{2}\alpha\frac{\phi\phi'}{r}\Big]\lambda\\
&+\Big[-2-\alpha+3\beta-3\sigma+(1+\alpha-5\beta+11\sigma)\frac{\phi}{2r}+(1+\alpha-\beta+\sigma)\phi'+(-\alpha+\beta-3\sigma)\frac{\phi\phi'}{2r}-\frac{5}{2}\sigma\frac{\phi^2}{r^2}+\frac{1}{2}\sigma\frac{\phi'\phi^2}{r^2}\Big]\mu\Bigg\}=0\,,
\end{aligned} \tag{28}$$

$$2(\lambda-\mu)-2r^2\mu_{rr}-r^2\nu_{rr}+2r(\lambda_r-3\mu_r-\nu_r)+2m_g^2r^2\Bigg\{-(3+3\alpha+\beta)+(1+2\alpha+\beta)(\phi'+2\frac{\phi}{r})-(\alpha+\beta)(\frac{\phi^2}{r^2}+2\frac{\phi'\phi}{r})$$

$$+\beta\frac{\phi'\phi^2}{r^2}+\Bigg[-\frac{1}{2}(1+3\alpha+3\beta+\sigma)+\frac{1}{2}(\alpha+2\beta+\sigma)(\phi'+2\frac{\phi}{r})-(\sigma+\beta)(\frac{\phi\phi'}{r}+\frac{\phi^2}{2r^2})+\frac{1}{2}\sigma\frac{\phi'\phi^2}{r^2}\Bigg]\nu$$

$$+\Bigg[-\frac{1}{2}(1+2\alpha+\beta)\phi'+(\alpha+\beta)\frac{\phi\phi'}{r}-\frac{1}{2}\beta\frac{\phi'\phi^2}{r^2}\Bigg]\lambda+\Bigg[-(1+2\alpha+\beta)\frac{\phi}{r}+(\alpha+\beta)(\frac{\phi'\phi}{r}+\frac{\phi^2}{r^2})-\beta\frac{\phi'\phi^2}{r^2}\Bigg]\mu\Bigg\}=0\,, \tag{29}$$

$$(\lambda-\mu)=r\mu_r+\frac{1}{2}r\nu_r+f(r)\,, \tag{30}$$

where $f(r)$ is an arbitrary function of r. The subscripts y and r in these relations represent partial differentiation with respect to y and r respectively. Prime in ϕ' denotes derivative with respect to r. In addition to the generalized field equations (17), the Bianchi identities lead to the constraint:

$$m_g^2\quad\nabla^A\,\overline{X}_{AB}\Bigg)=0\,, \tag{31}$$

where ∇^A denotes the covariant derivative with respect to physical metric g_{AB}. In the cases $m_g\neq 0$, the linearized form of these constraints for $B=1$ and $B=4$ are respectively as follows

$$2\alpha(\frac{\phi^2}{r^2}-\frac{\phi\phi'}{r})+2(1+\alpha)(\phi'-\frac{\phi}{r})+\Bigg[-\frac{1}{2}(1+2\alpha+\beta)+(\alpha+\beta)\frac{\phi}{r}-\frac{1}{2}\beta\frac{\phi^2}{r^2}\Bigg]r\nu_r+\Bigg[-(\alpha+1)\frac{\phi}{r}+\alpha\frac{\phi^2}{r^2}\Bigg]r\mu_r$$

$$+\Bigg[(\alpha+\beta)(\phi'-\frac{\phi}{r})+\beta(\frac{\phi^2}{r^2}-\frac{\phi\phi'}{r})\Bigg]\nu+\Bigg[(\alpha+1)(\frac{\phi}{r}-\phi')+2\alpha(\frac{\phi\phi'}{r}-\frac{\phi^2}{r^2})\Bigg]\mu=0\,, \tag{32}$$

$$\frac{\partial}{\partial y}(\overline{X}_{44})=\Bigg[-\frac{1}{2}(1+3\alpha+3\beta+\sigma)+\frac{1}{2}(\alpha+2\beta+\sigma)(\phi'+2\frac{\phi}{r})-(\sigma+\beta)(\frac{\phi\phi'}{r}+\frac{\phi^2}{2r^2})+\frac{1}{2}\sigma\frac{\phi'\phi^2}{r^2}\Bigg]\nu_y$$

$$+\Bigg[-\frac{1}{2}(1+2\alpha+\beta)\phi'+(\alpha+\beta)\frac{\phi\phi'}{r}-\frac{1}{2}\beta\frac{\phi'\phi^2}{r^2}\Bigg]\lambda_y+\Bigg[-(1+2\alpha+\beta)\frac{\phi}{r}+(\alpha+\beta)(\frac{\phi'\phi}{r}+\frac{\phi^2}{r^2})-\beta\frac{\phi'\phi^2}{r^2}\Bigg]\mu_y=0\,, \tag{33}$$

where other components of the constraint (31) are satisfied automatically. Contrary to the easy DGP model, which we studied in our previous paper [59], the presence of graviton mass terms in the 5D field equations (26)-(30) makes it more difficult to find an exact solution. The linearised form of the Israel-Darmois matching conditions (19) will lead to the following boundary conditions (on the brane) for the filed equations in the bulk

$$-\frac{1}{2}\Big(2\mu_y+\lambda_y\Big)|_{y=0^+}=r_c\Big[-\frac{1}{r^2}\Big(\mu-\lambda+3r\mu_r+r^2\mu_{rr}-r\lambda_r\Big)\Big]\,, \tag{34}$$

$$-\frac{1}{2}\Big(2\mu_y+\nu_y\Big)|_{y=0^+}=r_c\Big[-\frac{1}{r^2}\Big(\mu-\lambda+r\mu_r+r\nu_r\Big)\Big]\,, \tag{35}$$

$$-\frac{1}{2}\Big(\nu_y+\lambda_y+\mu_y\Big)|_{y=0^+}=r_c\Big[-\frac{1}{2r}\Big(r\nu_{rr}+r\mu_{rr}+2\mu_r+\nu_r-\lambda_r\Big)\Big]\,. \tag{36}$$

Note that these equations are hold on the brane outside our spherical object, where ρ and p are zero. Solving the linearized bulk field equations (26)-(30) with constraints (32) and

(33) (resulting from the Bianchi identities), is a very difficult task in non-unitary gauges. Therefore, here we consider some additional simplifying assumptions for the theory. The first assumption is that we find solutions in the unitary gauge, i.e. $\phi(r) = r$. In this gauge, the linearized form of all the higher order combinations, $\mathcal{U}_2$, $\mathcal{U}_3$ and $\mathcal{U}_4$ vanish such that

$$\overline{X}_{AB} = \mathcal{K}_{AB} - [\mathcal{K}]g_{AB} = \text{diag}\ \left((\frac{1}{2}\lambda + \mu), -(\frac{1}{2}\nu + \mu), -\frac{1}{2}r^2(\nu + \lambda + \mu), -\frac{1}{2}r^2 \sin^2(\theta)(\nu + \lambda + \mu), -\frac{1}{2}(\nu + \lambda + 2\mu) \right). \tag{37}$$

Therefore, in the unitary gauge, the free parameters of the theory are absent in the field equations and effectively the Fierz-Pauli mass term is rebuilt. In this situation, the equations that should be solved are simplified to the following system of partial differential equations

$$2(\mu - \lambda) + 2r^2\mu_{rr} + 6r\mu_r - 2r\lambda_r + r^2(\lambda_{yy} + 2\mu_{yy}) + m_g^2 r^2(\lambda + 2\mu)) = 0\,, \tag{38}$$

$$2(\lambda - \mu) - 2r\mu_r - 2r\nu_r - r^2(\nu_{yy} + 2\mu_{yy}) - m_g^2 r^2(\nu + 2\mu) = 0\,, \tag{39}$$

$$-r^2(\nu_{rr} + \mu_{rr}) - r\nu_r - 2r\mu_r + r\lambda_r - r(\nu_{yy} + \lambda_{yy} + \mu_{yy}) - m_g^2 r^2(\nu + \lambda + \mu) = 0\,, \tag{40}$$

$$2(\lambda - \mu) - 2r^2\mu_{rr} - r^2\nu_{rr} + 2r(\lambda_r - 3\mu_r - \nu_r) - m_g^2 r^2(\nu + \lambda + 2\mu) = 0\,, \tag{41}$$

$$(\lambda - \mu) = r\mu_r + \frac{1}{2}r\nu_r + f(r)\,, \tag{42}$$

where $f(r)$ is an arbitrary function. The constraint equations (32) and (33) in the unitary gauge are represented by the following equations

$$\nu_r + 2\mu_r = 0, \tag{43}$$

$$\nu_y + \lambda_y + 2\mu_y = 0. \tag{44}$$

The Israel-Darmois junction conditions on the brane are independent of the gauge and are the same as before, that is, Eqs. (34)-(36). The three free parameters of the theory α, β and σ do not exist in the unitary gauge. The general solution of the bulk field equations with the mentioned assumptions that satisfies the constraint equations are obtained as follows

$$\lambda = \mu = a\cos(m_g y) + b\sin(m_g y), \tag{45}$$

$$\nu = -3\mu = -3\Big(a\cos(m_g y) + b\sin(m_g y)\Big), \tag{46}$$

where a and b are integration constants. By putting these solutions into the Israel-Darmois junction conditions, we see that b should be zero. Therefore, the linearized theory in the unitary gauge leads to the following line element on the brane

$$ds_4{}^2 = -(1-3a)dt^2 + (1+a)dr^2 + r^2(1+a)d\Omega^2\,. \tag{47}$$

Actually, this solution after the coordinates redefinition $(t, r) \to (t', r')$, where $t' = (\sqrt{1-3a})\,t$ and $r' = (\sqrt{1+a})\,r$, reduces to the 4D flat Minkowski metric. But, this coordinates transformation leads to the appearance of the temporal component of the Stückelberg fields as

$$\phi^0 = (1-\eta)t', \qquad \eta = 1 - \frac{1}{\sqrt{1-3a}}, \tag{48}$$

and the scalar mode of massive graviton resulting from this Stückelberg field is $\pi = \frac{1}{2}\eta\Lambda^3 t'^2$. The final result is a flat 3-brane solution which is accompanied by the obtained scalar mode.

For the second simplifying assumption, we decided to work in non-unitary gauges. In this case, the free parameters of the theory (α, β, σ) play important roles in characterizing the properties of the solution, such as the (A)dS curvature scale. Moreover, the unknown scalar field $\phi(r)$ is coupled nonlinearly with other unknown metric components which can make the field equations more difficult to solve. We should determine a consistent scalar field $\phi(r)$ together with other unknown functions from field equations. Here, we consider three additional simplifying assumptions. The first one is to assume that the functional $\mu(r, y)$ be just a function of the extra dimension y, which in the non-unitary gauge $(\phi(r) \neq r)$ it could be a reasonable assumption. In this situation, solving the field equations could be slightly more easier. Moreover, we can restrict ourself to specific choices of the free parameters. The second assumption is to consider the case $\alpha = \beta = \sigma = 0$, which is equivalent to the choices $\alpha_3 = -\alpha_4 = \alpha_5 = -1$. By this assumption, the effective energy-momentum tensor $\overline{X}_{AB}$ takes the Fierz-Pauli structure, i.e. $\overline{X}_{AB} = \mathcal{K}_{AB} - [\mathcal{K}]g_{AB}$. However, the expression of it's components are not the same as eq. (37), which resulted in the unitary gauge. For this special choices of the free parameters, the components of $\overline{X}_{AB}$ takes the following form

$$\overline{X}_{00} = 3 - \phi' - 2\frac{\phi}{r} + (3 - \phi' - 2\frac{\phi}{r})\nu + \frac{1}{2}\phi'\lambda + \frac{\phi}{r}\mu\,, \tag{49}$$

$$\overline{X}_{11} = -2 + 2\frac{\phi}{r} - \frac{1}{2}\nu - \frac{\phi}{r}\mu + (-2 + 2\frac{\phi}{r})\lambda\,, \tag{50}$$

$$\overline{X}_{22} = r^2\Big(-2 + \frac{\phi}{r} + \phi' - \frac{1}{2}\nu - \frac{1}{2}\phi'\lambda + (-2 + \frac{\phi}{2r} + \phi')\mu\Big)\,, \tag{51}$$

$$\overline{X}_{33} = \sin^2(\theta)\overline{X}_{22}\,, \tag{52}$$

$$\overline{X}_{55} = -3 + 2\frac{\phi}{r} + \phi' - \frac{1}{2}\nu - \frac{1}{2}\phi'\lambda - \frac{\phi}{r}\mu\,, \tag{53}$$

where reduce to (37) for $\phi(r) = r$. The constraint equations (32) and (33) for these special choices of the parameters are

$$2(\phi' - \frac{\phi}{r}) - \frac{1}{2}r\nu_r - \phi\mu_r + (\frac{\phi}{r} - \phi')\mu = 0\,, \tag{54}$$

$$\nu_y + \phi'\lambda_y + 2\frac{\phi}{r}\mu_y = 0. \tag{55}$$

The scalar field $\phi(r)$ is yet stayed coupled with other fields which this makes finding the solutions of the field equations difficult. The third assumption we do is to linearize the field equations with respect to the scalar field by considering $\phi(r) \ll 1$ and ignoring the nonlinear terms in the above equations. Therefore, by imposing these three assumptions we reach to the following field equations that should be solved analytically

$$2(\mu - \lambda) - 2r\lambda_r + r^2(\lambda_{yy} + 2\mu_{yy}) + 2m_g^2 r^2(3 - \phi' - 2\frac{\phi}{r} + 3\nu) = 0\,, \tag{56}$$

$$2(\lambda - \mu) - 2r\nu_r - r^2(\nu_{yy} + 2\mu_{yy}) + 2m_g^2 r^2(-2 + 2\frac{\phi}{r} - \frac{1}{2}\nu - 2\lambda) = 0\,, \tag{57}$$

$$-r^2\nu_{rr} + r(\lambda_r - \nu_r) - r^2(\nu_{yy} + \lambda_{yy} + \mu_{yy}) + 2m_g^2 r^2(-2 + \frac{\phi}{r} + \phi' - \frac{1}{2}\nu - 2\mu) = 0\,, \tag{58}$$

$$2(\lambda-\mu)-r^2\nu_{rr}+2r(\lambda_r-\nu_r)+2m_g^2r^2(-3+\phi'+2\frac{\phi}{r}-\frac{1}{2}\nu)=0\,, \tag{59}$$

$$(\lambda-\mu)=\frac{1}{2}r\nu_r+f(r)\,, \tag{60}$$

$$2(\phi'-\frac{\phi}{r})-\frac{1}{2}r\nu_r=0\,, \tag{61}$$

$$\nu_y=0. \tag{62}$$

These equations are valid in the regions where ν, μ, λ and ϕ are very small. We obtained the following solutions for these linearized field equations

$$\nu(r,y)=a,$$

$$\lambda(r,y)=\mu(r,y)=\frac{3}{4}a,$$

$$\phi(r)=(1+a)r, \tag{63}$$

where a is an integration constant. Note that these solutions are valid in the regions where the obtained $\phi(r)$ is very small, i.e $r \ll \left(\frac{1}{1+a}\right)$. However, the metric here is accompanied by a nontrivial spatial backgrounds for the Stückelberg fields, $\pi^i = x^i - \phi^i = -ax^i, (i=1,2,3)$, where $x^i = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. The corresponding 4D line element on the 3-brane is given by

$$ds_4{}^2=-(1+a)dt^2+(1+\frac{3}{4}a)dr^2+r^2(1+\frac{3}{4}a)d\Omega^2\,. \tag{64}$$

However, in this case the solution on the brane transforms also to the 4D flat Minkowski metric, after the coordinates redefinition $(t,r) \to (t',r')$ with $t' = \sqrt{1+a}\,t$ and $r' = \sqrt{1+\frac{3}{4}a}\,r$. Due to this coordinates transformation, the temporal and spatial components of the Stückelberg fields will take the following forms

$$\phi'^0=\frac{1}{\sqrt{1+a}}\,t', \tag{65}$$

$$\phi'^i=\frac{1+a}{\sqrt{1+\frac{3}{4}a}}\,x'^i. \tag{66}$$

Finaly, the scalar mode of massive graviton resulting from these Stückelberg fields is

$$\pi=\frac{\Lambda^3}{2}(\delta\, t'^2+\gamma\, x'^2), \tag{67}$$

where the constants δ and γ are related to a via $\delta = 1-\frac{1}{\sqrt{1+a}}$ and $\gamma = 1-\frac{1+a}{\sqrt{1+\frac{3}{4}a}}$.

In this paper, we considered two simplified linear theories in both unitary and non-unitary gauges and found in both cases a flat solution on the brane with different background Stückelberg fields (after a coordinates redefinition). In non-unitary gauge, we restricted ourself to special choices of the free parameters of the theory. Finding a general analytical solution for the linear theory with arbitrary α, β and σ together with the unknown scalar field $\phi(r)$ and then screening the solution on the brane to be consistent with junction conditions is a very difficult and complicated procedure. However, in the regions where we should keep nonlinear terms in the field equations, solving them will be more intricate. In this situation, we can pursue alternative approaches, such as solving the equations numerically or finding the effective 4D field equations on the brane for the new braneworld massive gravity theory and then solving them analytically [60, 61]. We are working on these subjects and the outcomes after completion will be presented in another paper.

4 Summary

We know that a way in which a massive graviton can naturally arise is from higher dimensional scenarios, such as KK, ADD and DGP theories. It has been shown that there is a deep connection between the DGP braneworld gravity and massive gravity theories. The graviton in the DGP setup acquires effectively a soft mass and the induced gravity term in the brane action acts as a kinetic term for a 4D graviton, while the bulk Einstein-Hilbert term acts as a gauge invariant mass term. Studying massive gravity from extra dimensional point of view can be useful for better understanding of certain aspects of the dRGT massive gravity theory and its bigravity and multi-gravity extensions. This fact was the original motivation of this paper to construct an extension of massive gravity in the spirit of braneworld scenarios. We have constructed a combination of the braneworld scenario and covariant de Rham-Gabadadze-Tolley (dRGT) massive Gravity, where we suppose that the five-dimensional bulk graviton is massive. We considered a static 5D configuration with spherical symmetry on the brane, aimed at separately determining the effects of bulk graviton mass term and also the large extra dimension on the gravitational interactions on the brane. Then, by a detailed analytical treatment, the effects of the nonlinear ghost-free massive gravity on brane dynamics and effective gravitational potential on the brane are examined. In this manner, the full field equations and their weak field limits together with the generalized Israel-Darmois junction conditions on the brane are presented. This set of equations are so complicated to be solved analytically without some simplifying assumptions. For this reason, by adopting some simplifying assumptions, we were able to find two classes of four-dimensional spherically symmetric solutions on the brane in unitary and non-uniary gauges. Both of them were flat solutions on the brane with different background Stckelberg fields (after a coordinates redefinition). We note that general massive braneworld solutions should reduce to the massless braneworld solution in the limit of zero bulk graviton mass as has been studied in [59]. To restore the GR or the original DGP model on the brane, we should consider certain nonlinear terms in the bulk field equations and the brane junction conditions, which make the solving procedure more difficult (because of the bulk mass terms). We are attempting to follow alternative approaches, such as solving the field equations numerically or finding the effective 4D field equations on the brane for the new massive braneworld theory [60, 61], to examine the Vainshtein mechanism in our model. This issue in the absence of the Boulware-Deser ghost and also the instability issue are subject of our forthcoming work.

References

[1] Riess A. G., et al., 1998, Astron. J, vol. 116, no. 3, pp. 1009-1038.

[2] Perlmutter S., et al., 1999, Astron. J, vol. 517, no. 2, pp. 565-586.

[3] Spergel D. N., et al., 2007, Astrophys. J., Suppl. Ser., vol. 170, no. 2, pp. 377-466.

[4] Csaki C., 2004, "TASI Lectures on Extra Dimensions and Branes," *Shifman, M. (ed.) et al.: *From fields to strings*, vol. 2* pp. 967-1060, [arXiv:hep-ph/0404096].

[5] Maartens R. and Koyama K., 2010, Living. Rev. Relat., vol. 13, no. 5.

[6] Dvali G., Gabadadze G., and Porrati M., 2000, Phys. Lett. B, vol. 485, no. 1-3, pp. 208-214.

[7] Deffayet C., 2001, Phys. Lett. B, vol. 502, no. 1-4, pp. 199-208.

[8] Dvali G. R. and Gabadadze G., 2001, Phys. Rev. D, vol. 63, no. 6, p. 065007.

[9] Deffayet C., Dvali G. R., and Gabadadze G., 2002, Phys. Rev. D, vol. 65, no. 4, p. 044023.

[10] Dvali G. R., Gabadadze G., Kolanovic M., and Nitti F., 2001, Phys. Rev. D, vol. 64, no. 8, p. 084004.

[11] Abbott B. P., et al. [LIGO Scientific and Virgo Collaborations], 2016, Phys. Rev. Lett., vol. 116, no. 6, p. 061102.

[12] Abbott B. P., et al. [LIGO Scientific and Virgo Collaborations], 2016, Phys. Rev. Lett., vol. 116, no. 24, p. 241103.

[13] Abbott B. P., et al., [LIGO Scientific and Virgo Collaborations], 2016, Phys. Rev. Lett., vol. 116, no. 22, p. 221101.

[14] Fierz M. and Pauli W., 1939, Proc. R. Soc. Lond. Ser. A, vol. 173, no. 953, pp. 211-232.

[15] van Dam H. and Veltman M. J. G., 1970, Nucl. Phys. B, vol. 22, no, 2, pp. 397-411.

[16] Zakharov V. I., 1970, JETP. Lett., vol. 12, pp. 312-314.

[17] Vainshtein A. I., 1972, Phys. Lett. B, vol. 39, no. 3, pp. 393-394.

[18] Deffayet C., Dvali G., Gabadadze G., and Vainshtein A., 2002, Phys. Rev. D, vol. 65, no. 4, p. 044026.

[19] Porrati M., 2001, Phys. Lett. B, vol. 498, no. 1-2, pp. 92-96.

[20] Kogan I. I., Mouslopoulos S., and Papazoglou A. 2001, Phys. Lett. B, vol. 503, no. 1-2, pp. 173-180.

[21] Boulware D. G. and Deser S., 1972, Phys. Rev. D, vol. 6, no. 12, p. 3368.

[22] Arkani-Hamed N., Georgi H., and Schwartz M. D., 2003, Ann. Phys, vol. 305, no. 2, pp. 96-118.

[23] de Rham C. and Gabadadze G., 2010, Phys. Rev. D, vol. 82, no. 04, p. 044020.

[24] de Rham C., Gabadadze G., and Tolley A. J., 2011, Phys. Rev. D, vol. 106, no. 23, p. 231101.

[25] Hassan S. and Rosen R. A., 2012, Phys. Rev. Lett., vol. 108, no. 4, p. 041101.

[26] Hassan S., Rosen R. A., and Schmidt-May A., 2012, J. High. Energy. Phys., vol. 1202, p. 026.

[27] Hassan S. and Rosen R. A., 2012, J. High. Energy. Phys., vol. 1204, p. 123.

[28] de Rham C., Matas A., and Tolley A. J., 2014, Class. Quantum. Grav., vol. 31, no. 06, p. 065004.

[29] Gao X., 2014, Phys. Rev. D, vol. 90, no. 6, p. 064024.

[30] de Rham C., 2014, Living. Rev. Relat., vol. 17, no. 7.

[31] Hinterbichler K., 2012, Rev. Mod. Phys, vol. 84, no. 2, p. 671.

[32] Schmidt-May A. and von Strauss M., 2016, J. Phys. A: Math. Theor., vol. 49, no. 18, p. 183001.

[33] Koyama K., Niz G., and Tasinato G., 2011, Phys. Rev. Lett., vol. 107, no. 13, p. 131101.

[34] Koyama K., Niz G., and Tasinato G., 2011, Phys. Rev. D, vol. 84, no. 6, p. 064033.

[35] Nieuwenhuizen T., 2011, Phys. Rev. D, vol. 84, no. 2, p. 024038.

[36] de Rham C., Gabadadze G., Heisenberg L., and Pirtskhalava D., 2011, Phys. Rev. D, vol. 83, no. 10, p. 103516.

[37] D'Amico G., de Rham C. , Dubovsky S., Gabadadze G., Pirtskhalava D., and Tolley A. J., 2011, Phys. Rev. D, vol. 84, no. 12, p. 124046.

[38] Gumrukcuoglu A., Lin C., and Mukohyama S., 2011, J. Cosmol. Astropart., vol. 2011, p. 030.

[39] Arkani-Hamed N., Cohen A. G., and Georgi H., 2011, Phy. Rev. Lett., vol. 86, no. 21, p. 4757.

[40] Arkani-Hamed N., Cohen A. G., and Georgi H., 2001, Phy. Lett. B, vol. 513, no. 1-2, pp. 232-240.

[41] Gruzinov A., 2005, New. Astron., vol. 10, no. 4, pp. 311-314.

[42] Porrati M., 2002, Phys. Lett. B, vol. 534, no. 1-4, pp. 209-215.

[43] Luty M. A., Porrati M., and Rattazzi R., 2003, J. High. Energy. Phys., vol. 0309, p. 029.

[44] Nicolis A., and Rattazzi R., 2004, J. High. Energy. Phys., vol. 0406, p. 059.

[45] Koyama K., 2005, Phys. Rev. D, vol. 72, no. 12, p. 123511.

[46] Nicolis A., Rattazzi R., and Trincherini E., 2009, Phys. Rev. D, vol. 79, no. 6, p. 064036.

[47] Gabadadze G., 2009, Phys. Lett. B, vol. 681, no. 1, pp. 89-95.

[48] de Rham C. , 2010, Phys. Lett. B, vol. 688, no. 2-3, pp. 137-141.

[49] Hassan S. F. and Rosen R. A., 2012, J. High. Energy. Phys., vol. 1202, p. 126.

[50] Hinterbichler K. and Rosen R. A., 2012, J. High. Energy. Phys., vol. 1207, p. 047.

[51] de Rham C. and Gabadadze G., 2010, Phys. Lett. B, vol. 693, no. 3, pp. 334-338.

[52] Randall L. and Sundrum R., 1999, Phys. Rev. Lett., vol. 83, no. 17, p. 3370.

[53] Chacko Z., Graesser M. L., Grojean C., and Pilo L., 2004, Phys. Rev. D, vol. 70, no. 8, p. 084028.

[54] Gherghetta T., Peloso M., and Poppitz E., 2005, Phys. Rev. D, vol. 72, no. 10, p. 104003.

[55] Kakushadze Z., 2014, Acta. Phys. Pol. B, vol. 45, pp. 1671-1699.

[56] Kluson J., 2011, Class. Quantum. Grav., vol. 28, no. 15, p. 155014.

[57] Iglesias A. and Kakushadze Z., 2010, Phys. Rev. D, vol. 82, no. 12, p. 124001.

[58] Iglesias A. and Kakushadze Z., 2011, Phys. Rev. D, vol. 84, no. 8, p. 084005.

[59] Nozari K. and Asaiyan A., 2011, Class. Quantum. Grav., vol. 28, no. 12, p. 125017.

[60] Shiromizu T., Maeda K., and Sasaki M., 2000, Phys. Rev. D, vol. 62, no. 2, p. 024012.

[61] Maeda K-i., Mizuno Sh., and Torii T., 2003, Phys. Rev. D, vol. 68, no. 2, p. 024033.

[62] Do T. Q., 2016, Phys. Rev. D, vol. 93, no. 10, p. 104003.

[63] Dadhich N., Maartens R., Papadopoulos P., and Rezania V., 2000, Phys. Lett. B, vol. 487, no. 1-2, pp. 1-6.

[64] Kofinas G., Papantonopoulos E., and Zamarias V., 2002, Phys. Rev. D, vol. 66, no. 10, p. 104028.

[65] Middleton C. and Siopsis G., 2004, Mod. Phys. Lett. A, vol. 19, no. 30, p. 2259.

[66] Gregory R., 2009, Lect. Notes. Phys., vol. 769. pp. 259-298.

[67] Berezhiani L., Chkareuli G., de Rham C., Gabadadze G., and Tolley A. J., 2012, Phys. Rev. D, vol. 85, no. 4, p. 044024.

[68] Babichev E. and Brito R., 2015, Class. Quantum. Grav., vol. 32, no. 15, p. 154001.

[69] Tasinato G., Koyama K., and Niz G., 2013, Class. Quantum. Grav., vol. 30, no. 18, p. 184002.

[70] Babichev E. and Fabbri A., 2014, J. High. Energy. Phys., vol. 1407, p. 016.

[71] Babichev E. and Fabbri A. , 2014, Phys. Rev. D, vol. 90, no. 8, p. 084019.

[72] Cai R. G., Hu Y. P., Pan Q. Y., and Zhang Y. L., 2015, Phys. Rev. D, vol. 91, no. 2, p. 024032.

[73] Xu J., Cao L. M., and Hu Y. P., 2015, Phys. Rev. D, vol. 91, no. 12, p. 124033.

[74] Zhou Z., Wu J. P., and Ling Y., 2015, J. High. Energy. Phys., vol. 1508, p. 067.

[75] Hendi S. H., Eslam Panah B., and Panahiyan S., 2015, J. High. Energy. Phys., vol. 1511, p. 157.

[76] Ghosh S. G., Tannukij L., and Wongjun P., 2016, Eur. Phys. J. C, vol. 76, no. 3, p. 119.

[77] Do T. Q., 2016, Phys. Rev. D, vol. 94, no. 4, p. 044022.

[78] El-Menoufi B. K. and Sorbo L., 2015, Phys. Rev. D, vol. 91, no. 6, p. 064023.