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**Letter**

# On Equivalence of Definition of the Energy-Momentum Tensor in General Relativity and Field Theory

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**Abstract**. This work investigates a fundamental pedagogical question concerning the definition of the energy-momentum tensor in theoretical physics. Students learn the definition of energy-momentum tensor in the field theory and on the other hand a different definition of energy-momentum tensor in general relativity. why are these definitions equivalent? We present a proof establishing the equivalence between two seemingly distinct definitions: we analyze how the canonical energy-momentum tensor, obtained via Noether's theorem, corresponds to the symmetric energy-momentum tensor derived through variational methods in general relativity. Our approach involves a general coordinate transformation in which both the fields and the metric are varied simultaneously. By applying Noether's theorem and requiring the total variation of the action to vanish, we bridge the gap between the field-theoretic and geometric formulations. This analysis elucidates the deep conceptual link between the two definitions and enhances the understanding of energy-momentum in modern theoretical physics.

*Keywords*: General Relativity, Classical Field Theory, Energy-Momentum Tensor.

## 1 The definition of Energy momentum tensor in general relativity

Starting from the Einstein-Hilbert action, the Einstein field equation can be obtained by varying the action with respect to the metric. This action consists of geometrical part as the field and the matter part,

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}d^4x + \int \mathcal{L}_M\sqrt{-g}d^4x, \tag{1}$$

where for simplicity, we exclude the surface terms for the geometric part [1]. From the variation of action with respect to the metric (i.e. $\delta S/\delta g_{\mu\nu} = 0$), we get the Einstein tensor from the first term of the action and the energy-momentum tensor of the matter from the second term [2]. This leads to

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}, \tag{2}$$

where we set $c = 1$ and

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g_{\mu\nu}}. \tag{3}$$

In the variation process, we perform a diffeomorphic map of $\mathcal{M} \to \mathcal{M}'$ which involves the infinitesimal variation of metric, $g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$ ( with $\delta g_{\mu\nu} \ll 1$) around the stationary solution. We can simplify equation (3) by taking the variation in parts as

$$T^{\mu\nu} = -2\frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}} + g^{\mu\nu}\mathcal{L}_M. \tag{4}$$

This equation is the classical definition of the energy-momentum tensor in general relativity in curved space-time. In order to derive the energy-momentum tensor of a field in the Minkowski space one can apply the covariant principle, replacing $g_{\mu\nu} \to \eta_{\mu\nu}$ and $\nabla_\mu \to \partial_\mu$.

## 2 Definition of the energy-momentum tensor in classical field theory

In this section, we review the definition of the energy-momentum tensor in the classical field theory. Using the Neother theorem [3], let us consider a Lagrangian density of $\mathcal{L}_M(\psi^\ell, \partial\psi^\ell)$ where superscript $\ell$ represents any arbitrary class of fields. Now, we define a generic transformation of the field as

$$\psi^\ell(x) \to \psi^\ell(x) + i\epsilon(x)\mathcal{F}^\ell(\psi^\ell, \partial\psi^\ell), \tag{5}$$

where $\epsilon(x)$ is an infinitesimal function of spacetime and $\mathcal{F}^\ell$ contains the $\psi^\ell$ and derivatives of the field . Under this transformation, the action of $S = \int \mathcal{L}_M(\psi^\ell, \partial\psi^\ell)d^4x$ varies as

$$\delta S_M = i\int \left(\frac{\partial \mathcal{L}}{\partial \psi^\ell}\epsilon(x)\mathcal{F}^\ell + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\ell)}\partial_\mu(\mathcal{F}^\ell\epsilon(x))\right)d^4x. \tag{6}$$

The variation of action, taking $\epsilon$ as an infinitesimal parameter , which satisfies Euler-Lagrange equation is zero,

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi^\ell}\epsilon\mathcal{F}^\ell + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\ell)}\epsilon\partial_\mu(\mathcal{F}^\ell) = 0, \tag{7}$$

then, the variation of action considering $\epsilon$ as the function of space-time simplifies to

$$\delta S_M = i\int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\ell)}\mathcal{F}^\ell\partial_\mu\epsilon(x)d^4x, \tag{8}$$

where we can define a current as

$$J^\mu = -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\ell)}\mathcal{F}^\ell, \tag{9}$$

and taking derivatives in part in the equation (8), the current satisfies the conservation of charge as $\partial_\mu J^\mu = 0$, having $\delta S_M = 0$.

Now, we follow a similar argument to find symmetries of a matter field living on a Riemannian manifold, with a subclass of variation of the fields by an infinitesimal coordinate transformation of

$$x'^\mu = x^\mu + \epsilon^\mu(x). \tag{10}$$

According to this transformation, using the Taylor expansion, the field varies as

$$\psi^\ell(x+\epsilon) = \psi^\ell(x) + \psi^\ell_{;\nu}\epsilon^\nu(x). \tag{11}$$

Note that since we are doing the Taylor expansion of a field and at the same time to keep the variation of action covariant. Here ";" represents the covariant derivative. The action of the matter field by this transformation varies as

$$\delta S_M = \int \left( \frac{\partial \mathcal{L}_M}{\partial \psi^\ell} \psi^\ell_{;\nu} \epsilon^\nu(x) + \frac{\partial \mathcal{L}_M}{\partial (\psi^\ell_{;\lambda})} (\psi^\ell_{;\nu} \epsilon^\nu(x))_{;\lambda} \right) \sqrt{-g} d^4x. \tag{12}$$

We can rewrite this equation as

$$\delta S_M = \int \left( ( \frac{\partial \mathcal{L}_M}{\partial \psi^\ell} \psi^\ell_{;\nu} \epsilon^\nu(x) + \frac{\delta \mathcal{L}_M}{\partial \psi^\ell_{;\lambda}} (\psi^\ell_{;\nu})_{;\lambda} \epsilon^\nu(x) + \frac{\partial \mathcal{L}_M}{\partial \psi^\ell_{;\lambda}} \psi^\ell_{;\nu} \epsilon^\nu{}_{;\lambda}(x) \right) \sqrt{-g} d^4x, \tag{13}$$

noting that the first and the second terms of integration of (13) is $\epsilon^\nu(x) d\mathcal{L}_M/dx^\nu$. So, the integral simplifies to

$$\delta S_M = \int \left( \frac{d\mathcal{L}_M}{dx^\nu} \epsilon^\nu(x) + \frac{\partial \mathcal{L}_M}{\partial \psi^\ell_{;\lambda}} \psi^\ell_{;\nu} \epsilon^\nu{}_{;\lambda}(x) \right) \sqrt{-g} d^4x, \tag{14}$$

where differentiating by part and ignoring the surface term, the final result is

$$\delta S_M = \int \left( -\mathcal{L}_M \delta^\lambda{}_\nu + \frac{\partial \mathcal{L}_M}{\partial \psi^\ell_{;\lambda}} \psi^\ell_{;\nu} \right) \epsilon^\nu{}_{;\lambda}(x) \sqrt{-g} d^4x. \tag{15}$$

We note that this equation is similar to (8) and from the definition of Noether current, the definition of the energy-momentum tensor of this field is given by

$$T^\lambda{}_\nu = -\mathcal{L}_M \delta^\lambda{}_\nu + \frac{\partial \mathcal{L}_M}{\partial \psi^\ell_{;\lambda}} \psi^\ell_{;\nu}, \tag{16}$$

where imposing $\delta S_M = 0$ and integrating by parts of equation (15) results in the conventional form of conservation of energy-momentum tensor as $T^\lambda{}_{\nu;\lambda} = 0$. We note that in this variation process, we keep the space-time on the manifold unchanged and let the variation of the fields under an infinitesimal coordinate transformation.

# 3 On the equivalence of definition of energy momentum-tensor

In this section, our aim is to show the equivalence of the definition of energy-momentum tensor from the variation principle in general relativity and from the Neother theorem.

Let us take the Lagrangian of matter field as a function of field and metric as an individual field as $\mathcal{L}_M(\psi^\ell, \partial\psi^\ell, g_{\mu\nu})$. Then, we let the action of this Lagrangian,

$$S = \int \mathcal{L}_M(\psi^\ell, \partial\psi^\ell, g_{\mu\nu}) \sqrt{-g} d^4x,$$

vary just as a classical field, including the metric field as a rank-two tensor as well as the matter field, $\psi^\ell$. Under the transformation of $x^\mu \to x^\mu + \epsilon^\mu(x)$, we have both the variation of the field as $\delta\psi^\ell$ and the variation of metric as $\delta g^{\mu\nu}$. The total variation of action is

$$\delta S_M = \delta S_M^{(\psi)} + \delta S_M^{(g)}, \tag{17}$$

$$\delta S_M \quad = \quad \int \left(\frac{\delta \mathcal{L}}{\delta \psi^\ell}\delta\psi^\ell + \frac{1}{\sqrt{-g}}\frac{(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}\delta g_{\mu\nu}\right)\sqrt{-g}d^4x. \tag{18}$$

The second term by definition in general relativity is the energy-momentum tensor. So, we can rewrite this equation as

$$\delta S_M = \int \left(\frac{\delta \mathcal{L}}{\delta \psi^\ell}\delta\psi^\ell - \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}\right)\sqrt{-g}d^4x. \tag{19}$$

Since the field equations satisfy the Euler-Lagrange equations, then $\delta S_M = 0$.

In this first term of this equation, we substitute the variation of the action with respect to the field from equation (15) while splitting the second term into two symmetric parts. Also, we lower the index of $\epsilon^\mu(x)$ using the metric. Then, action can be written as

$$\int \left(-\frac{1}{2}\mathcal{L}_M g^{\mu\nu}(\epsilon_{\nu;\mu}(x) + \epsilon_{\mu;\nu}(x)) + \frac{1}{2}\frac{\partial \mathcal{L}_M}{\partial \psi^\ell_{,\mu}}\psi^\ell_{,\nu}\epsilon^\nu{}_{;\mu}(x) + \frac{1}{2}\frac{\partial \mathcal{L}_M}{\partial \psi^\ell_{,\nu}}\psi^\ell_{,\mu}\epsilon^\mu{}_{;\nu}(x) - \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}\right)\sqrt{-g}d^4x = 0. \tag{20}$$

The coordinate transformation as the diffeomorphic transformation results in variation of metric as

$$\delta g_{\mu\nu} = \epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}. \tag{21}$$

Substituting in equation (20), we can write the first term of this equation in terms of the variation of metric. For calculating the second and third terms, let us assume a generic kinetic term for the Lagrangian as $\mathcal{K} = f(-\frac{1}{2}\psi^\ell_{;\nu}\psi^\ell_{;\mu}g^{\mu\nu})$, then the variation of Lagrangian with respect to the fields, results in

$$\frac{\partial \mathcal{L}_M}{\partial \psi^\ell_{;\nu}} = \frac{\partial \mathcal{K}}{\partial \psi^\ell_{;\nu}} = -\mathcal{L}' g^{\mu\nu}\psi^\ell_{;\mu}, \tag{22}$$

where $\mathcal{L}'$ is the derivation of the Lagrangian with respect to the argument containing the kinetic term. Substituting in equation (20), the variation of action can be written as a combination of variation of metric and coordinate as follows

$$\frac{1}{2}\int \left(-\mathcal{L}_M g^{\mu\nu}\delta g_{\mu\nu} - \mathcal{L}'\nabla^\nu(\psi^\ell)\nabla^\mu\psi^\ell\left(\epsilon_{\nu;\mu}(x) + \epsilon_{\mu;\nu}(x)\right) - T^{\mu\nu}\delta g_{\mu\nu}\right)\sqrt{-g}d^4x = 0. \tag{23}$$

On the other hand, for a Lagrangian with a generic kinetic term, we vary with respect to the metric. The result is

$$\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = -\frac{1}{2}\mathcal{L}'\nabla^\mu(\psi^\ell)\nabla^\nu(\psi^\ell)\delta g_{\mu\nu}, \tag{24}$$

where substituting the identity of (21) in equation (24),

$$\mathcal{L}'\nabla^\mu(\psi^\ell)\nabla^\nu(\psi^\ell)(\epsilon_{\mu;\nu}(x) + \epsilon_{\nu;\mu}(x)) = \mathcal{L}'\nabla^\mu(\psi^\ell)\nabla^\nu(\psi^\ell)\delta g_{\mu\nu} = -2\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}\delta g_{\mu\nu}, \tag{25}$$

and substituting in equation (23), the variation of the action simplifies to

$$\frac{1}{2}\int \left(-\mathcal{L}_M g^{\mu\nu} + 2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + T^{\mu\nu}\delta g_{\mu\nu}\right)\delta g_{\mu\nu}\sqrt{-g}d^4x = 0, \tag{26}$$

and from this equation the energy-momentum tensor obtain as

$$T^{\mu\nu} = -2\frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}} + g^{\mu\nu}\mathcal{L}_M. \tag{27}$$

We note that this is identical to the definition of energy-momentum tensor from variation of metric which has been obtained by the variation of field via a genetic coordinate transformation.

## 4 conclusion

In this pedagogical work, we introduced the definition of the energy-momentum tensor in the conventional formalism of least action in general relativity by varying the action with respect to the metric. On the other hand, we introduced the energy-momentum tensor in the classical field theory where the Noether current is obtained from the action.

In order to show the equivalence of these two different definitions, we assumed that the Lagrangian of the matter is made of the matter field and the rank-two metric field. We applied the least action principle for this Lagrangian, taking into account that the two distinct fields for the matter ($\psi^\ell$ and $g_{\mu\nu}$) vary independently, and from the least action principle the variation of the action under infinitesimal coordinate transformation set to zero. Then we used the identities from the two different definitions of the energy-momentum tensor and showed that the two definitions of the energy-momentum tensor in general relativity and classical field theory are equivalent.

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The author contributed to data analysis, drafting, and revising of the paper and agreed to be responsible for all aspects of this work.

## Data Availability

No data available.

## Conflicts of Interest

The author declares that there is no conflict of interest.

## Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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