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Research Paper

Vacuum Stress between Conducting Plates for Neumann's Condition

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Abstract. In [1], we have found that the curved spacetime version of the vacuum stress between conducting plates was first done in flat spacetime by Brown and Maclay. Here, we calculate the energy-momentum tensor for the Casimir effect of parallel plates under *Neumann boundary conditions* to support our recent results found in [2]. We use direct calculation of energy-momentum tensor by employing the well understood point splitting method of regularization, and we show that the there is first order correction to the Casimir energy in curved spacetime proportional to $\frac{GM}{c^2R}$ where R and M are the radius and the mass of the source of gravity.

Keywords: Gavitational vacuum energy, Casimir effect, Cosmological constant problem, Quantum vacuum

1 Introduction

The energy-momentum tensor has a crucial role in quantum field theory in curved spacetime [3]. This is partially because of the fact that the notion of a particle (and energy) in curved spacetime is not as clear as that in flat spacetime. In fact, in the lack of global symmetries in curved spacetime, local quantities such as energy-momentum tensor are of crucial importance as they are locally describe the energy through a tensorial form. The energy momentum tensor for extended bodies have been extensively studied in classical systems. See [4] and references therein. In the quantum side, or more exactly the semi-classical side, there are few studies which concerns calculating the energy momentum tensor of an extended body other than a black hole. One of quantum energy momentum tensors which have been investigated in a curved background is the Casimir effect of parallel plates [5–8]. The importance of the subject is related to an old question that whether the quantum vacuum can describe the origin of the cosmological constant or not? There are many investigations which concern the possible relationship between the zero point energy of a quantum field to the cosmological constant appears in Einstein field equations [10]. The discrepancy between the obtained zero point energy and the current estimated cosmological constant is called the cosmological constant problem (CCP) [11,12].

In fact, the reaction of a gravitational system to the quantum vacuum of Casimir effect can be a good example to seek the answer to the question. The author has shown [2,9] that the Casimir energy of parallel plates has a first order correction proportional to $\frac{GM}{c^2R}$ which is many orders of magnitude greater than previous results obtained by others. In [2], we found such a result using a different approach of quasi-local stress tensor formalism. In this work, by direct calculation of the energy-momentum tensor, we provide strong support to the result under the influence of *Neumann boundary condition (NBC).* From electromagnetic theory we know that the electric field which is the derivative of the potential (a scalar field) vanishes on a metal boundary. Therefore, although the related calculations for the NBC are typically more complicated, the NBC sounds more physical than the Dirichlet one for the case of the scalar field.

The structure of the paper is as follows. In section 2, we review the point splitting method first introduced by Christensen [13]. In section 3, we apply the Neumann B.C. and find the suitable wave function for the scalar field as well as the non-vanishing energy momentum tensor components. In section 4, the energy has been found for the static spacetime we have considered in this work. Conclusion is the final section.

2 The energy-momentum tensor

According to Christensen [13], after employing the point-splitting method of regularization, the energy-momentum tensor for a scalar field in arbitrary spacetime is given by

$$\begin{aligned}\langle T_{\mu\nu}\rangle &= \lim_{x'\to x}\Bigg[\frac{(1-2\xi)}{4}\Big(G^{(1)}_{;\mu'\nu}+G^{(1)}_{;\mu\nu'}\Big)+\Big(\xi-\frac{1}{4}\Big)g_{\mu\nu}G^{(1)\ \sigma'}_{;\sigma}\\ &- \frac{\xi}{2}\Big(G^{(1)}_{;\mu\nu}+G^{(1)}_{;\mu'\nu'}\Big)+\frac{\xi}{8}g_{\mu\nu}\Big(G^{(1)\ \sigma}_{;\sigma}+G^{(1)\ \sigma'}_{;\sigma'}\Big)+\frac{\xi}{2}G_{\mu\nu}G^{(1)}\\ &+ \frac{3}{4}\xi^2 R g_{\mu\nu}G^{(1)}+\frac{3\xi-1}{4}m^2 g_{\mu\nu}G^{(1)}\Bigg],\end{aligned} \tag{1}$$

where

$$G^{(1)}(x,x')=\langle[\phi(x),\phi(x')]_+\rangle=2Im\ G_F, \tag{2}$$

is the Hadamard function and μ' denotes differentiation with respect to x'. G_F is the Feynmann function which satisfies the Klein-Gordon equation

$$(\Box-\xi R)G_F(x,\acute{x})=-\frac{\delta(x,\acute{x})}{\sqrt{-g}}. \tag{3}$$

We assume that the Casimir parallel plates are separated by small distance l. Thus, it can be shown [8] that any general static spacetime can be expanded in the space between the plates as follows

$$ds^2=(1+2\gamma_0+2\lambda_0 z)dt^2-(1+2\gamma_1+2\lambda_1 z)\left(dx^2+dy^2+dz^2\right), \tag{4}$$

in which $\gamma_0=-\gamma_1=-\frac{Gm}{c^2R}<<1,\ \lambda_0 z=-\lambda_1 z=\frac{Gm}{c^2R^2}z<<1$ [8].

The general form of the symmetric green function G_F has been found to be [1]

$$G(z,\acute{z})=\frac{Y_1(z)Y_2(\acute{z})}{W(\acute{z})p_0(\acute{z})},\qquad z<\acute{z}, \tag{5}$$

in which

$$Y(z)=D_0\left(1-(\frac{\lambda}{2}+\frac{a}{4b})z\right)\sin\left(\sqrt{b}z(1+\frac{a}{4b}z)+\Theta_0\right). \tag{6}$$

The Θ_0 should be found by imposition of boundary conditions and D_0 from commutation relations on wave function. Note that for $z>\acute{z}$ it suffices to do the interchange $z\leftrightarrow\acute{z}$ in the nominator of the equation (5). However, at the limit $\acute{z}\to z$ we do not need the part of the Green function for $z>\acute{z}$.

3 The Neumann boundary conditions

The Neumann boundary condition on plates at $z = 0, z = l$ is defined by

$$\partial_z G_F(z, z')|_{z=0,l} = 0, \tag{7}$$

which in turn gives

$$\partial_z Y_1(0) = 0, \quad \partial_z Y_2(l) = 0, \tag{8}$$

on account of (5).

Upon imposition of (8) on (6) we find

$$Y_1(z) = \left(1 - (\frac{\lambda}{2} + \frac{a}{4b})z\right) \sin\left(S(z) + 2\delta_0\right), \qquad z < \acute{z}, \tag{9a}$$

$$Y_2(z) = \left(1 - (\frac{\lambda}{2} + \frac{a}{4b})z\right) \sin\left(S(z) - S(l) + 2\delta_0\right), \qquad \acute{z} < z, \tag{9b}$$

in which

$$S(z) = \sqrt{b}\left(z + \frac{a}{4b}z^2\right), \ \delta_0 = \frac{\epsilon}{\sqrt{b}} + \frac{\pi}{2}. \tag{10}$$

We keep only those terms which are within second order perturbations in terms of the parameters $\lambda_0, \lambda_1, \gamma_0, \gamma_1$. Therefore, the Green function is found to be

$\mathbf{z} < \acute{\mathbf{z}}$:

$$\mathbf{g}_F(z, \acute{z}) = \frac{1 - \gamma_0 - \gamma_1 - \lambda(z + z')}{2\sqrt{b}\sin(\sqrt{b}l)} \Bigg\{ - \cos(\sqrt{b}\alpha) - \cos(\sqrt{b}\beta) + \frac{a}{4\sqrt{b}} \times \\ \left((z^2 - \acute{z}^2 + l^2)\sin(\sqrt{b}\beta) + (z^2 + \acute{z}^2 - l^2)\sin(\sqrt{b}\alpha) + l^2\cot(\sqrt{b}l)\cos(\sqrt{b}\alpha)\right)\Bigg\}, \tag{11}$$

where

$$\alpha = z + \acute{z} \quad -l, \quad \beta = z - \acute{z} + l = \Delta z + l, \tag{12}$$

and

$$G_F(x, \acute{x}) \quad = \int \tfrac{d\omega dk_\perp}{(2\pi)^3} \mathbf{g}_F(z, \acute{z}) e^{-i\omega(t-\acute{t}) + \vec{k}_\perp.(\vec{x} - \acute{\vec{x}})}. \tag{13}$$

Using the equations (1), (2), and (13), after a very long and tricky calculation [1] we find the $T_{\mu\nu}$ components as follows:

$$\begin{aligned} \langle T_{00}\rangle = & \frac{E_0}{l}\left(1 + 2\gamma_0 - 4\gamma_1 + \frac{2}{5}\lambda_0(3l - z) - \frac{2}{5}\lambda_1(3l + 4z)\right) + \frac{c_0}{(\Delta z)^4} \\ & + \frac{B}{90\pi^2}\Big[+ 8zA_1(\alpha) - l^2A_2(\alpha) + (2z^2 - l^2)A_3(\alpha) - 5A_4(\alpha)\Big] \\ & + \frac{1}{12\pi^2}(\xi - \frac{1}{6})\Big[- 6\left(1 + 2\gamma_0 - 4\gamma_1 - 2\lambda_1 z\right)A_1(\alpha) - Bl^2A_2(\alpha) + (2z^2 - l^2)B\ A_3(\alpha) \\ & + 2(4\lambda_1 + 5\lambda_0)A_4(\alpha)\Big], \end{aligned} \tag{14}$$

$$\begin{aligned}\langle T_{11}\rangle = \langle T_{22}\rangle = &-\left(1-2\gamma_1-\frac{2}{5}\lambda_0(2z-l)-\frac{2}{5}\lambda_1(3z+l)\right)\frac{E_0}{l}+\frac{c_1}{(\Delta z)^4}\\ &-\frac{B}{180\,\pi^2}\Big[-8zA_1(\alpha)+l^2A_2(\alpha)-(2z^2-l^2)A_3(\alpha)-5A_4(\alpha)\Big]\\ &+\frac{1}{12\pi^2}(\xi-\frac{1}{6})\Big[+6\,(1-2\gamma_1-2\lambda_0 z)\,A_1(\alpha)+Bl^2A_2(\alpha)\\ &-(2z^2-l^2)BA_3(\alpha)-2(2\lambda_0+7\lambda_1)A_4(\alpha)\Big],\end{aligned} \tag{15}$$

$$\begin{aligned}\langle T_{33}\rangle =&\frac{3E_0}{l}\left(1-2\gamma_1-\frac{2}{3}(2\lambda_0+\lambda_1)z+\frac{2}{3}(\lambda_0-\lambda_1)l\right)+\frac{c_2}{(\Delta z)^4}\\ &+(\xi-\frac{1}{6})\Big[\frac{1}{4\pi^2}(\lambda_0+2\lambda_1)A_4(\alpha)\Big],\end{aligned} \tag{16}$$

where

$$c_0 = -\frac{1}{2\pi^2}\Big[1+2\gamma_0-4\gamma_1-\frac{2}{5}(\lambda_0+4\lambda_1)z\Big], \tag{17}$$

$$c_1 = \frac{1}{2\pi^2}\left(1-2\gamma_1-\frac{2}{5}(2\lambda_0+3\lambda_1)z\right), \tag{18}$$

$$c_2 = -\frac{3}{2\pi^2}\left(1-2\gamma_1-\frac{2}{3}(2\lambda_0+\lambda_1)z\right), \tag{19}$$

and

$$A_1(\alpha) = \int_0^\infty \frac{\kappa^3\cosh(\kappa\alpha)}{\sinh\kappa l}d\kappa, \tag{20a}$$

$$A_2(\alpha) = \int_0^\infty \frac{\kappa^4\cosh(\kappa\alpha)\cosh(\kappa l)}{\sinh^2\kappa l}d\kappa, \tag{20b}$$

$$A_3(\alpha) = \int_0^\infty \frac{\kappa^4\sinh(\kappa\alpha)}{\sinh\kappa l}d\kappa, \tag{20c}$$

$$A_4(\alpha) = \int_0^\infty \frac{\kappa^2\sinh(\kappa\alpha)}{\sinh\kappa l}d\kappa, \tag{20d}$$

and $E_0 = -\pi^2/1440l^4$ is the Casimir energy in flat spacetime. For a brief review of the main steps led to the above results see appendix A.

4 The Energy and Pressure

The volume energy in a static spacetime is given by [2]

$$E = \int <0|T^0{}_\nu|0> \zeta^\nu\sqrt{-g}d^3x, \tag{21}$$

in which ζ^μ is the corresponding Killing vector. We show that only the first line of (14) contributes the energy upon integration on z. By assuming $\zeta^\mu = \delta^0_\mu$ for a typical static spacetime, we first consider the second line of (14) which can be written as

$$\begin{aligned} E =& A\int_0^l (1-\gamma_0+3\gamma_1+(3\lambda_1-\lambda_0)z)\,\langle T_{00}\rangle \\ =& \frac{E_0}{l}A\int_0^l\left(1+\gamma_0-\gamma_1+(3\lambda_1-\lambda_0)z+\frac{2}{5}\lambda_0(3l-z)-\frac{2}{5}\lambda_1(3l+4z)\right)+\frac{c_0}{(\Delta z)^4} \\ &+\frac{B}{90\pi^2}\int_0^l\Big[+8zA_1(\alpha)-l^2A_2(\alpha)+(2z^2-l^2)A_3(\alpha)-5A_4(\alpha)\Big] \\ +&\frac{1}{12\pi^2}(\xi-\frac{1}{6})\int_0^l\Big[-6\,(1+2\gamma_0-4\gamma_1-2\lambda_1 z)\,A_1(\alpha)-Bl^2A_2(\alpha)+(2z^2-l^2)B\;A_3(\alpha) \\ &+2(4\lambda_1+5\lambda_0)A_4(\alpha)\Big]. \end{aligned} \tag{22}$$

Now, using (12) and (20a)-(20d) we can easily find

$$\int_0^l A_1(\alpha)dz=\int_0^\infty\frac{\kappa^3}{\sinh\kappa l}[\int_0^l\cosh(\kappa\alpha)dz]d\kappa=0, \tag{23a}$$

$$\int_0^l zA_1(\alpha)dz=\int_0^\infty\frac{\kappa^3}{\sinh\kappa l}[\int_0^l z\cosh(\kappa\alpha)dz]d\kappa=\frac{l}{2}\int_0^\infty\kappa^2, \tag{23b}$$

$$\int_0^l A_2(\alpha)dz=l^2\int_0^\infty\kappa^3 d\kappa+\frac{\pi^4}{120l^2}, \tag{23c}$$

$$\int_0^l A_3(\alpha)dz=\int_0^\infty\frac{\kappa^4}{\sinh\kappa l}[\int_0^l\sinh(\kappa\alpha)dz]d\kappa=0, \tag{23d}$$

$$\int_0^l z^2A_3(\alpha)dz=\int_0^\infty\frac{\kappa^4}{\sinh\kappa l}[\int_0^l z^2\sinh(\kappa\alpha)dz]d\kappa=\frac{l}{2}\int_0^\infty\kappa^2+\frac{l^2}{2}\int_0^\infty\kappa^3+\frac{\pi^4}{240l^2}, \tag{23e}$$

$$\int_0^l A_4(\alpha)dz=0. \tag{23f}$$

By substituting these results back into (22), we end up with

$$E=A\left(1+\gamma_0-\gamma_1+\frac{\lambda_0-\lambda_1}{2}l\right)E_0. \tag{24}$$

A point should be noted here. The divergent terms such as $\frac{1}{3(\Delta z)^n}$ are the typical side effects of the point separation method [14]. Therefore. in the course of integration over z, they must be dropped away as they are boundary effects. The result (24) is exactly what we found in [2] using the quasi-local approach to the quantum vacuum energy in curved spacetime.

For pressure, we find the projection of $<0|T_{\mu\nu}|0>$ on $u^\mu=(\sqrt{-g_{33}})^{-1}\delta_3^\mu$ and find

$$P=u^3u^3<0|T_{33}|0>=A\left(1-4\gamma_1-\frac{4}{3}(\lambda_0+2\lambda_1)z+\frac{2(\lambda_0-\lambda_1)}{3}l\right). \tag{25}$$

5 Concluding Remarks

In this paper, we found strong support for our previous work [2] by direct calculation of the energy momentum tensor for a scalar field confined in the space between two material plates under Neumann boundary condition. The energy momentum tensor in equations (14)-(16) was found to be the same as that in [1] for Dirichlet boundary condition except for some sign change in the divergent part of the $T_{\mu\nu}$, i.e. in the α-dependent part . Therefore, we have shown that the finite part of the energy is the same for both Dirichlet and Neumann boundary conditions. Also, we shown that the divergent part vanishes upon the integration over z.

As the obtained $T_{\mu\nu}$ shows, the flat spacetime limit (or $\lambda_0 = \lambda_1 = \gamma_0 = \gamma_1 = 0$) is completely finite in the case of conformal coupling of the field, the effect that is previously well known in the literature. Another point is that the only divergent part of the $T_{\mu\nu}$ is $\frac{1}{(\Delta z)^4}$ for both flat and curved spacetime. This is also in agreement to the results have been found in [14]. Thus, the curved spacetime does not alters the structure of the divergencies relative to that of the flat spacetime.

Since we have obtained the volume energy-momentum tensor, and that there is a finite part of the energy resides exactly on the surfaces [15], finding probable divergent terms other than $\frac{1}{(\Delta z)^4}$ will be postponed to a more comprehensive study which involves the surface part of the energy momentum tensor.

The lowest order correction to the energy was $\gamma_0 - \gamma_1$, which does not in general vanish, and shows a first order correction to the Casimir energy. For example, in a general weak static spacetime of the form

$$ds^2 = (1 + 2\phi(x))dt^2 - (1 - 2\phi(x))d\overrightarrow{x}^2, \;\; \phi(x) << 1, \tag{26}$$

we have $\gamma_0 - \gamma_1 = 2\phi(R)$ where R is the distance the plates are located at relative to the center of the source of gravitational field. In fact, $\phi(R)$ is the lowest order perturbation expansion of $\phi(x)$:

$$\phi(x) = \phi(R) + \frac{d\phi(x)}{dx}|_{x=R}(x - R) + \cdots, \;\; (x - R) << 1. \tag{27}$$

Appendix A: brief review of the calculations

Using equations (11), it can be shown that

$$\mathbf{g}_F = \frac{1}{2}(N - M), \tag{28a}$$

$$M = (1 - \gamma_0 - \gamma_1)\frac{\cos(\sqrt{b}\alpha)}{\sqrt{b}\sin(\sqrt{b}l)} - \frac{al^2}{4b}\frac{\cos(\sqrt{b}\alpha)\cos(\sqrt{b}l)}{\sin^2(\sqrt{b}l)} - \left(\frac{a}{4\sqrt{b}}(z^2 + z'^2 - l^2) + \frac{2\epsilon}{\sqrt{b}}\right)\frac{\sin(\sqrt{b}\alpha)}{\sqrt{b}\sin(\sqrt{b}l)} - 2\epsilon z\frac{\cos(\sqrt{b}\alpha)}{\sqrt{b}\sin(\sqrt{b}l)} \tag{28b}$$

$$N = -(1 - \gamma_0 - \gamma_1)\frac{\cos(\sqrt{b}\beta)}{\sqrt{b}\sin(\sqrt{b}l)} + \frac{al^2}{4b}\frac{\cos(\sqrt{b}\beta)\cos(\sqrt{b}l)}{\sin^2(\sqrt{b}l)} + \frac{a}{4\sqrt{b}}(z^2 - z'^2 + l^2)\frac{\sin(\sqrt{b}\beta)}{\sqrt{b}\sin(\sqrt{b}l)} + 2\epsilon z\frac{\cos(\sqrt{b}\beta)}{\sqrt{b}\sin(\sqrt{b}l)}, \tag{28c}$$

$$\epsilon = \frac{\lambda}{2} + \frac{a}{4b}. \tag{28d}$$

The changes

$$\begin{aligned} &\sqrt{b}\alpha \to \sqrt{b}\alpha + \pi, \\ &\frac{a}{4\sqrt{b}}(z^2 + z'^2 - l^2) \to \frac{a}{4\sqrt{b}}(z^2 + z'^2 - l^2) + \frac{2\epsilon}{\sqrt{b}}, \end{aligned} \tag{29}$$

are the main differences in the course of calculations compared to that of the Dirichlet boundary conditions.

Having the above relations in hand, we can compute the energy-momentum tensor components. For instance, the $\langle T_{00}\rangle$ component is given by [1]

$$\begin{aligned}\langle T_{00}\rangle = \frac{1}{6}\lim_{z'\to z} Im\int \frac{d\omega dk_\perp}{(2\pi)^3}\Big[\frac{9}{2}\omega^2 - \frac{3}{2}\frac{g_{00}}{g_{11}}k_\perp^2 + \frac{g_{00}}{4g_{11}}\Big(\partial_z^2 + \partial_{z'}^2 - 4\partial_z\partial_{z'}\Big)\\ + \frac{3\lambda_0-\lambda_1}{4}(\partial_z+\partial_{z'})\Big]\mathbf{g}_F\\ + (\xi - \frac{1}{6})\lim_{z'\to z} Im\int \frac{d\omega dk_\perp}{(2\pi)^3}\Big[\frac{3}{2}\omega^2 + \frac{3}{2}\frac{g_{00}}{g_{11}}k_\perp^2 + \frac{g_{00}}{4g_{11}}\Big((\partial_z^2 + \partial_{z'}^2) + 8\partial_z\partial_{z'}\Big)\\ + \frac{3\lambda_0-\lambda_1}{4}(\partial_z+\partial_{z'})\Big]\mathbf{g}_F.\end{aligned} \tag{30}$$

Authors' Contributions

The author contributed to data analysis, drafting, and revising of the paper and agreed to be responsible for all aspects of this work.

Data Availability

No data available.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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