Iranian Journal of Astronomy and Astrophysics
Vol. 11, No. 1, Winter 2024, 73–80
©Available online at http://ijaa.du.ac.ir
DOI: 10.22128/ijaa.2024.755.1174
Online ISSN: 2383–403X

Iranian Journal of Astronomy and Astrophysics

Research Paper

The Cigar-Shaped Background Metric with Supersymmetry Approaches and Thermal Properties of System

Omid Jalili*[1] · Shahin Toni[2] · Jafar Sadeghi[3]

[1] Department of Physics, Islamic Azad University-Nour Branch, P.O.BOX 4641738373, Nour, Mazandaran, Iran;
*email: omid_jalili@yahoo.com

[2] Department of Physics, Islamic Azad University-Nour Branch, P.O.BOX 4641738373, Nour, Mazandaran, Iran;
email: shahin.toni75@gmail.com

[3] Department of physics, Faculty of Science, Mazandaran University, P.O.BOX 47415-416, Babolsar, Mazandaran, Iran;
Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O.BOX 19395-5531, Tehran, Iran;
email: pouriya@ipm.ir

Received: 28 November 2023; **Accepted:** 3 July 2024; **Published:** 11 July 2024

Abstract. In this paper, we introduce the cigar- shaped solution. Such metric can be obtained by the Schwarzschild black hole. Here, we write the Dirac equation in the corresponding background. We take some change of variable for the above equation and re-write the mentioned equation similar to the hypergeometric equation. We take advantage from the known polynomial (hypergeometric) and factorized the Dirac equation in cigar- shaped background in terms of first order equations and achieve the wave function. Also, this first order equations lead us to use super symmetry approach and obtain the energy spectrum. Finally, we take information from thermodynamic and investigate the thermal properties of the system.

Keywords: Cigar-Shaped metric, Supersymmetry, Thermal properties

1 Introduction

The Schrödinger effort to find the relativistic version of his equation led him to Kline-Gordon equation where could be able to explain the spectrum of the Lyman, Blamer etc. series but could not explain the Paschen's spectrum. The Schrödinger himself think that the neglecting of spin was the origin of this discrepancy [1]. The Dirac tried to make a relativistic wave equation with spin but he started to remove the negative probability of the theory. According to the Dirac the origin of the negative probability of the Kline-Gordon equation was the presence of second order time derivative. Hence, he introduced his famous relativist wave equation that was first order in both space and time variables, which is given by

$$i\gamma^a \partial_a \psi - m\psi = 0. \tag{1}$$

*Corresponding author

The Dirac equation explain all of known spectrum series of hydrogen' atom. It take into account the spin and have not the negative probability and more it predict the anti particle! But, what is the energy spectrum in curved space? To answer this, we need to generalize the Dirac equation to curved space. The original Dirac equation is expressed in flat space (Minkowski space). In order to extend Dirac equation to curved space, we need to back some general system where the topology getting more important. Since the form of Dirac equation is covariant thus it is sufficient to replace equation (1) with the following expression

$$i\gamma^{\mu}\nabla_{\mu}\psi - m\psi = 0. \tag{2}$$

The gama matrix in general coordinate is defined as $\gamma^{\mu} \equiv \partial x^{\mu}/\partial y^{a}\gamma^{a}$ where y^{a} is the coordinate in locally Minkowski space and the x^{μ} is the general coordinate. Namely, the gama matrix is transformed as a vector field where its value in a locally Minkowski space is the Dirac matrix. The transformation coefficients is named tetrad and denoted by $e_{a}^{\mu} = \partial x^{\mu}/\partial y^{a}$. The γ^{μ} matrixes satisfy the anti commutation relation $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$ where $g^{\mu\nu}$ is the background metric. Because the the metric is related to Minkowski metric by the relation $g^{\mu\nu} = e_{a}^{\mu}e_{b}^{\nu}\eta^{ab}$; thus, we can express the connection as a function of tetrad and then the covariant derivatives will be function of tetrad. Finally, one can write the Dirac equation as [2],

$$[i\gamma^{a}e_{a}^{\mu}\partial_{\mu} + \frac{i}{2}\gamma^{a}\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}e_{a}^{\mu}) - m]\psi = 0. \tag{3}$$

We are going to obtain the energy spectrum of the Dirac particles(Fermions) in a well known background as cigar-shaped background. The cigar-shaped background metric first introduced by E. Witten in the relation of string theory and black holes [3]. We express the (1+1) Dirac equation in cigar-shaped background and then solve it by the supersymmetry method. The organization of this paper is as follows. In section 2, we discuss cigar-shaped background metric. In section 3, we obtain the energy spectrum of a Fermion in such a background by supersymmetry method. By the energy spectrum of system at hand, we obtain the partition function and other thermodynamics properties in section 4. In section 5, we have some conclusion and future suggestions.

2 Dirac Equation in Cigar-Shaped Background Metric

As we know, the quantum theory in curved space-time including two dimensional gravity play important role in cosmology and string theory. Here, we consider the curved space-time metric which is as cigar background which is given by,

$$ds^2 = \tanh^2 x dt^2 - dx^2, \quad x \geq 0. \tag{4}$$

We note here the scaler curvature is given by,

$$R = 4\,\mathrm{sech}^2\, x, \tag{5}$$

where in limit of $x \to 0$ the corresponding scalar curvature has not any singulary. So, one can say that in above mentioned limit the curvature is regular. The cigar-shaped metric can be obtained by the known Schwarzschild metric; so, Schwarzschild metric is,

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{6}$$

By using the following change of variable,

$$u = \frac{(\acute{u} + \acute{v})}{2}, \quad v = \frac{(\acute{u} - \acute{v})}{2}, \tag{7}$$

where

$$\acute{u} = (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh(\frac{t}{4M}) \tag{8}$$

$$\acute{v} = (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh(\frac{t}{4M}), \tag{9}$$

we can rewrite the equation (6) as,

$$ds^2 = \frac{32M^3 e^{\frac{-r}{2M}}}{r} dudv + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{10}$$

By using suitable transformation to above equation we will arrive at [2]

$$ds^2 = \frac{1}{1-uv} dudv. \tag{11}$$

Such background metric is singular at $uv = 1$. If we take the following change of variable,

$$u = -\frac{1}{2} e^{\acute{x}-t}, \quad v = \frac{1}{2} e^{\acute{x}+t}, \quad \acute{x} = x + \ln(1 - e^{-2x}), \tag{12}$$

and put the equation (11), we will have cigar-shaped metric as (4). Here, we consider the metric (4) and rewrite the Dirac equation as,

$$[i\omega \coth x + \gamma^0 \gamma^1 (\partial_x + \frac{1}{\sinh 2x}) + im\gamma^0]\psi = 0. \tag{13}$$

Then, if we take the separation of variables

$$\psi(x,t) = e^{-i\omega t} \psi(x) = e^{-i\omega t} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \tag{14}$$

then, the corresponding Dirac equation will be as,

$$(-i\omega \coth x + \partial_x + \frac{1}{\sinh 2x})\psi_1 + im\psi_2 = 0, \tag{15}$$

$$(-i\omega \coth x - \partial_x - \frac{1}{\sinh 2x})\psi_1 + im\psi_1 = 0. \tag{16}$$

We take the following change of variable in equation (15) and (16)

$$\psi_2 = \tanh^{-\frac{1}{2}}(x) \varphi_2(x). \tag{17}$$

Then, one can rewrite the equation (15) and (16) as,

$$y(1-y)\phi_2''(y) + (\frac{1}{2} - y)\phi_2'(y) - (\frac{k^2}{4} - \frac{\beta'(\beta'-1)}{4y})\varphi_2 = 0, \tag{18}$$

where $\beta' = -i\omega$ and $k^2 = \beta'^2 + m^2$. Now, we are going to take the equation (18) and obtain the corresponding solution. In that case, we employ the supersymmetry approaches and study the equation (18).

3 Supersymmetry Approach and Cigar-Shaped Background Metric

The idea of supersymmetry first introduced in relativistic quantum mechanics. In 1976, Nicolai applied its procedure to non-relativistic quantum mechanics. In 1981, E. Witten use it for one dimensional model. After that the supersymmetry was a major tool in quantum mechanics and many other fields of physics [14,15]. Here, we apply such an approch to our problem [5–8,16]. In order to solve the equation (18) and obtain some supersymmetry information, we take the change of variable as

$$\varphi(y) = f(y)p(y). \tag{19}$$

We put this in equation (18), we will arrive the following equation,

$$\begin{aligned} & y(1-y)F''(y) - \Big[2y(1-y)\frac{p'(y)}{p(y)} + (\frac{1}{2} - y)\Big]F'(y) \\ & + \Big[y(1-y)\frac{p''(y)}{p(y)} + (\frac{1}{2} - y)\frac{p'(y)}{p(y)} - \frac{k^2}{4} + \frac{\beta'(\beta'-1)}{4y}\Big]F(y) = 0. \end{aligned} \tag{20}$$

We compare equation (20) with the following equation [4]

$$y(y-1)F''_n(\alpha,\eta)(y) + [(\alpha+1)-(\alpha+\eta+2)(y)]F'_n(\alpha,\eta)(y) + n(n+\alpha+\eta+1)F_n^{(\alpha,\eta)}(y) = 0, \tag{21}$$

then, we will obtain the $p(y)$ as,

$$p(y) = y^{\frac{1}{2}(\alpha+\frac{1}{2})}(1-y)^{\frac{1}{2}(\eta+\frac{1}{2})}. \tag{22}$$

So, the general wave function will be as,

$$\varphi(y) = Ny^{\frac{1}{2}(\alpha+\frac{1}{2})}(1-y)^{\frac{1}{2}(\eta+\frac{1}{2})}F_n^{(\alpha,\eta)}(y). \tag{23}$$

If we compare the third part of equation (20), one can arrange the energy spectrum or $k^2 = m^2 + \beta'^2$ as,

$$k^2 = 4n(n+\alpha+\eta+1) + (\alpha+\eta+1)^2. \tag{24}$$

So, the corresponding m^2 will be as

$$m^2 = E = -\beta'^2 + 4n(n+\alpha+\eta+1) - (\alpha+\eta+1)^2, \tag{25}$$

$$E = E_0 + 4n(n+\alpha+\eta+1), \tag{26}$$

where

$$E_0 = \beta'^2 + (\alpha+\eta+1)^2. \tag{27}$$

Now, we are going to look at the system in supersymmetry point of view. In order to investigate such topic, one can factorize the equation (18) in terms of first order equation which is given by [4],

$$A_n(y) = y(1-y)\frac{d}{dy} + ny - \frac{n(n+\eta)}{2n+\alpha+\eta}, \tag{28}$$

$$B_n(y) = -y(1-y)\frac{d}{dy} + (n+\alpha+\eta)y - \frac{(n+\alpha)(n+\alpha+\eta)}{2n+\alpha+\eta}. \tag{29}$$

Here, one can say that the two first order operators (28) and (29) are satisfied by the following equations

$$B_n(y)A_n(y)\varphi_n(y) = \varphi_n(y), \tag{30}$$

$$A_n(y)B_n(y)\varphi_{n-1}(y) = \epsilon_n\varphi_{n-1}(y). \tag{31}$$

Also, the equation (30) and (31) with (28) and (29) lead us to the following results,

$$B_n(y)\varphi_{n-1}(y) = \varphi_n(y), \tag{32}$$

$$A_n(y)\varphi_n(y) = \epsilon_n\varphi_{n-1}(y), \tag{33}$$

where

$$\epsilon_n = \frac{(n+\alpha)(n+\eta)}{(2n+\alpha+\eta)^2}[n(n+\alpha+\eta)]. \tag{34}$$

The obtained results of energy give us opportunity to investigate the thermal properties of the system.

4 Thermal Properties of Corresponding System

So, we are going to study the thermal properties of the system [9–13]. In order to investigate the thermal properties of such system we have to consider the above mentioned energy which is given by

$$E = E_0 + 4n(n+\alpha+\eta+1). \tag{35}$$

Meanwhile, we have following expression $\beta'^2 = (\alpha+\eta+1)^2$. In any physical system if we want to study the thermal properties as phase transition, critical point, entropy, internal energy and free energy we shall need to have the corresponding partition function. In order to obtain the partition function for the energy (35), one can write the general formula as,

$$Z = \sum_{n=0}^{\infty} e^{-\beta E_n}, \tag{36}$$

where $\beta = \frac{1}{KT}$. Now, we consider the energy as mentioned before in equation (35). We put equation (35) into equation (36), we arrive at

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-\beta(E_0+(4n(n+\alpha+\eta+1))} \\ &= e^{-\beta E_0}\sum_{n=0}^{\infty} e^{-\beta(4n(n+\alpha+\eta+1))} \\ &= e^{-\beta E_0}\sum_{n=0}^{\infty} e^{-\beta(4n(n+\alpha+\eta+1))}. \end{aligned} \tag{37}$$

Now, we use some mathematical remark and change in equation (37) the form of summation to integral. So, we have following expression

$$\begin{aligned} Z &= e^{-\beta E_0}\int_0^{\infty} e^{-4\beta n^2-4\beta(\alpha+\eta+1)n}dn \\ &= e^{-\beta E_0}\int_0^{\infty} e^{-4\beta(n+\frac{1}{2}(\alpha+\eta+1)^2}e^{\beta(\alpha+\eta+1)^2}dn \\ &= e^{-\beta E_0+\beta(\alpha+\eta+1)^2}\int_0^{\infty} e^{-4\beta(n+\frac{1}{2}(\alpha+\eta+1)^2}dn. \end{aligned} \tag{38}$$

Here, we take $y = n + \frac{1}{2}(\alpha + \eta + 1)^2$; so,

$$Z = e^{-\beta E_0 + \beta(\alpha+\eta+1)^2} \int_0^\infty e^{-4\beta y^2} dy. \tag{39}$$

Since, the integrand is an even function, we can rewrite the integral as

$$Z = e^{-\beta E_0 + \beta(\alpha+\eta+1)^2} \frac{1}{2} \int_{-\infty}^\infty e^{-4\beta y^2} dy. \tag{40}$$

Thus, the partition function will be as,

$$Z = \frac{\sqrt{\pi}}{4} (\frac{1}{\beta})^{\frac{1}{2}} e^{\beta[(\alpha+\eta+1)^2 - E_0]}. \tag{41}$$

By using the above partition function, one can obtain all thermal quantities of the system. The internal energy will be as

$$U = -\frac{\partial}{\partial\beta} \ln Z = \frac{1}{2\beta} - \beta[(\alpha+\eta+1)^2 - E_0] = \frac{1}{2}KT - \frac{1}{KT}[(\alpha+\eta+1)^2 - E_0]. \tag{42}$$

As we know, the particle in the cigar-shaped background has three terms in energy U. If we have flat space the energy U just has the $\frac{1}{2}KT + E_0\beta$. But the term of $-\beta[(\alpha+\eta+1)^2]$ is coming from non-flat space. As we can see the internal energy increase with temperature and have a non-zero energy at temperature zero. Now, we are also going to calculate the Helmholtz free energy as

$$A = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta}(\ln \frac{\sqrt{\pi}}{4} - \frac{1}{2} \ln \beta + \beta[(\alpha+\eta+1)^2 - E_0]). \tag{43}$$

Here, we have also four terms. The most important term here is $-\frac{1}{\beta} \ln \frac{\sqrt{\pi}}{4} + \frac{1}{2} \ln \beta$. Because such term will be a correction. If we look at any black hole, we will see some correction term which is complectly agree with literatures. As we know, the calculation of entropy is very important in black holes and some metric back ground. Because such quantity help us to consider the black as a bit of some information which is similar to computer science. On the other hand, we have several approach to calculating the entropy of system. So, here we also obtain the entropy which is give by,

$$S = -\frac{\partial}{\partial T} A = -\frac{\partial A}{\partial \beta} \frac{\partial \beta}{\partial T} = \frac{1}{KT^2} \frac{\partial A}{\partial \beta} = \frac{4k\beta^2}{\sqrt{\pi}} - \frac{k\beta}{2} + k\beta^2[(\alpha+\eta+1)^2 - E_0]. \tag{44}$$

Also, for the entropy we have some correction term which play important role in quantum gravity.
Again the entropy will increase with temperature as in flat space.

5 Conclusions

In this paper, we have solved the Dirac equation in a certain space time by super symmetry approach. We have discussed the appropriate wave function in the cigar-shaped space. The energy spectrum of the particle is derived. This energy function depends on the geometrical parameter of the background. Finally, we look advantage from the obtained energy and investigated the thermal properties of systems. Here, by using the partition function, we obtained the internal energy U, Helmoltz free energy and entropy of corresponding back ground. The results from thermal quantities such as U, A, and S shown that we had some correction term, which is important in quantum gravity.

Authors' Contributions

All authors have the same contribution.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare no potential conflicts of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or nonprofit sectors.

References

[1] Weinberg, S., 2005, The Quantum Theory of Fields, Cambridge university press.

[2] Sinha, A., & Roychoudhury, R., 1994, Int. J. Theor. Phys., 33, 1511.

[3] Witten, E., 1991, Phys. Rev. D, 44, 315.

[4] Sadeghi, J., & Moayyeri, H., 2007, Int. J. Theor. Phys., 46, 3115.

[5] Sadeghi, J., 2006, EPJ. B, 50, 453.

[6] Fakhri, H., & Sadeghi, J., 2004, Mod. Phys. Lett. A, 19, 615.

[7] Fakhri, H., & Sadeghi, Int. J. Theor. Phys., 43, 457.

[8] Sadeghi, J., 2007, Int. J. Theor. Phys., 46, 492.

[9] Sadeghi, J., & Farahani, H., 2014, Int. J. Theor. Phys., 53, 3683, [arXiv:1308.1054].

[10] Sadeghi, J. Pourhassan, B., & Asadi, A., 2013, EPJ. C, 74, 2680, [arXiv:1209.1235].

[11] Jafarzade, Kh., & Sadeghi, J., 2017, Int. J. Theor. Phys., 56, 3387 .

[12] Soroushfar, S., Pourhassan, B., & Sakall, I., 2024, Physics of the Dark Universe, 44.

[13] Pourbagher, A., & Amani, A., 2019, Ap&SS, 364.

[14] Cooper, F., Khare, A., & Sukhatme, U., 2001, Supersymmetry in Quantum Mechanics, World Scientific, Singapore.

[15] Junker G., 1996, Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin.

[16] Jafarizadeh, M. A., & Fakhri, H., 1998, Annals of Phys., 266, 178.