Iranian Journal of Astronomy and Astrophysics
Vol. 10, No. 4, Autumn 2023, 303–319
©Available online at http://ijaa.du.ac.ir
DOI: 10.22128/ijaa.2023.706.1152
Online ISSN: 2383–403X

Iranian Journal of Astronomy and Astrophysics

Research Paper

Thermodynamics of Horndeski Black Holes with Generalized Uncertainty Principle

Mohaddeseh Seifi*[1] · Akram S. Sefiedgar[2]

[1] Department of Theoretical Physics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, P.O. Box 47416–95447, Iran;
*email: m.seifi@stu.umz.ac.ir

[2] Department of Theoretical Physics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, P.O. Box 47416–95447, Iran;
email: a.sefiedgar@umz.ac.ir

Received: 12 August 2023; **Accepted:** 19 December 2023; **Published:** 25 December 2023

Abstract. Horndeski theory is the most general scalar-tensor extension of General Relativity with second order field equations. It may be interesting to study the effects of the Generalized Uncertainty Principle on a static and asymptotically flat shift symmetric solutions of the Horndeski black holes. With this motivation, here we obtain the modified black hole temperatures in shift symmetric Horndeski gravity by employing the Generalized Uncertainty Principle. Using the corrected temperature, the entropy and heat capacity are calculated with details. We also investigate the tunneling probability of particles from Horndeski black holes horizon and possible correlations between the emitted modes (particles).

Keywords: Generalized Uncertainty Principle, Quantum Gravity, Horndeski Theory, Black Hole thermodynamics

1 Introduction

At present, it is believed that Nature can be described by quantum mechanics and general relativity. In 1915, Albert Einstein proposed General Relativity (GR) that is able to successfully describe physical phenomena in astrophysics and cosmology [1,2]. Besides all the significant achievements, general relativity can not describe some theoretical and observational issues. One of the known failures of GR is the problem of black hole singularities that leads to various black hole spacetimes singularities. It seems that GR is cursed with own solutions, i.e., black holes. Also, the cosmological constant problem and the issue of the dark matter/energy can not be explained by GR (for more details see for instance [3–5]). During the recent decades, many efforts have been undertaken to construct a more comprehensive theory. One way to modify GR is reconstructing the geometric part of the Einstein field equations [6,7]. A special class, that is, the most general scalar-tensor theory with second order field equations, proposed by Horndeski in 1970s [8]. Recently, Hondeski theory has been received much attention and investigated in astrophysics and cosmology [9–14]. More attractively, Horndeski black holes have been investigated. For instance, spherically symmetric and static solutions [15–17], black hole solutions in the presence of a cosmological

---
* *Corresponding author*

constant and magnetic field [18,19], the observational results and gravitational lensing effects for Horndeski black holes [20–22] are studied. Moreover, thermodynamics of Horndeski black holes are studied, Hawking temperature and entropy and circular orbits are investigated also in [23–26].

Trying to construct a quantum theory of gravity leads to a minimal measurable length of the order of the Planck length, $\ell_P \sim 10^{-35}m$. Most quantum gravity approaches such as string theory [27–30], loop quantum gravity [31] and quantum geometry [32] predict the existence of a minimal measurable length in spacetime [33]. Also, the existence of a minimal measurable length can be supported from micro-black hole Gedanken experiment [34]. In GUP concept, the Heisenberg Uncertainty Principle (HUP) is modified to the so called Generalized Uncertainty Principle (GUP) [35–42]. Incorporation of GUP effects in standard quantum mechanics' problems reveals several novel corrections and modifies the results in high energy regime [43–47]. On the other hand, Doubly Special Relativity (DSR) proposes an upper bound for a test particle's momentum [48–50]. In fact, the existence of a minimal measurable length restricts a test particle's momentum to take a maximal measurable momentum of the order of the Planck momentum [51–53]. Several interesting and novel results are obtained by considering both a minimal length and a maximal momentum [40,54–56]. Moreover, when one considers curvature effects, it can be shown that there is a nonzero minimal uncertainty in momentum measurement too [37,38]. That is, in large distances, where the curvature of space time becomes important, momentum cannot be precisely determined. With the path integral formulation, such noncommutative background geometries can ultraviolet and infrared regularize quantum field theories in arbitrary dimensions through minimal uncertainties both in positions and in momenta (for more details, see [37,38]). It is important to note that natural cutoffs are essentially related to the compactness of corresponding symplectic manifold [57]. It is well-known that thermodynamic quantities of a black hole can be obtained by the standard uncertainty principle. So, in this respect incorporation of the GUP can modify the black hole physics. Recently, black holes, as a connection between general relativity and quantum mechanics, have been investigated widely in GUP framework. For instance, the GUP prevents black holes from total evaporation. Also the GUP modifies Hawking temperature [58]. So, because of the existence of a maximal temperature originating from minimal length/maximal energy, the GUP predicts a non-radiating remnant of the order of the Planck mass in the final stage of evaporation. So, while it provide a possible candidate for dark matter [59], it may be also a clue for solving the black hole information loss problem and interestingly opens a realistic door for studying the final stage of black hole evaporation [60–66]. The importance of the subject lies in the fact that black holes are essentially a quantum gravity object and therefore GUP as a phenomenological aspect of quantum gravity provides a more realistic framework to study black hole physics and thermodynamics. This feature has shown its efficiency in recent years study of black hole physics.

Although the effects of GUP on the thermodynamics and Hawking radiation of a large number of black holes has been studied in literature, the effect of GUP on the thermodynamics of Horndeski black hole apparently has not attracted attention these years. By recent advances in Optics of Black Holes via shadow cast of black holes, such as supermassive black holes attributed to M87 and Sgr A*, it has been opened new windows on the viability of alternative gravitational theories in one hand and constraining these theories through observations on the other hand. Therefore, the corrections into Horndeski black hole's metric via GUP, essentially provides a tool for constraining the most general scalar-tensor theory of gravity, Horndeski theory; the issue that has been considered for other types of gravities and their black hole solutions in the seminal work of Vagnozzi et. al. [67]. In addition, since Horndeski gravity is the most general scalar-tensor theory of gravity, it is possible in principle to find traces of these GUP induced modifications in experiments that will be designed for testing the role of the scalar degree of freedom. For instance, the status of the equivalence principle in Horndeski gravity is in principle a tool to check the role of the scalar degree and may it be capable to give some hints about the tiny quantum gravitational effects encoded in the GUP. It is also important to say that the existence of a scalar degree of freedom and its derivatives, minimally or non-mibimally coupled to gravity in the gravitational action, potentially mimics a more realistic black hole system in real world. In fact, the black holes are objects in a cosmological background which in late time is positively accelerating due to dark energy/modified gravity. So, it is essentially more reliable to treat black holes in scalar-tensor theories than the more theoretically simplified cases such as the Schwarzschild black hole. Albeit, we are aware that this analogy is not perfect since dark energy is evolving and here, we are dealing with a scalar field out of cosmological dynamics. But, this study essentially has something to do with these deep concepts. These are actually some reasons for reliability of such a study.

With this motivations, we adopt Horndeski gravity with phenomenological quantum gravitational effects to study the black hole thermodynamics. We consider a generalized/extended uncertainty relation that includes a minimal length, a minimal momentum and a maximal momentum to modify the black hole temperature and entropy. Then, by using the modified black hole temperature, we obtain the modified heat capacity. Finally, we consider Parikh-Wilczek tunneling process to describe the Hawking radiation emitted from a Horndeski black hole. We compute tunneling rate and also possible correlation between emitted modes (particles). To be more clarified, we study possible correlations between the emitted particles, a feature that can be used by itself to address at least a part of the lost information in the process of black hole formation. The motivation for performing such a study in Horndeski framework is the existence of a gap in this respect in literature in one side, and the fact that Horndeski theory is the most general scalar-tensor theory of gravity where incorporation of quantum gravitational effects may bring new physics in the realm of black hole thermodynamics in this framework on the other side.

2 Horndeski Theory

In the modern formulation of the Horndeski gravity, the action takes the following form [17]

$$S = \int \sqrt{-g} d^4x \left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_4^{bH} + \mathcal{L}_5^{bH}\right), \tag{1}$$

where $g \equiv det(g_{\mu\nu})$ and $g_{\mu\nu}$ is the metric tensor. In our case, we investigate a class of the Horndeski theory which posses shift symmetry, $\phi \longrightarrow \phi + constant$,. It includes six arbitrary functions of the scalar field and its canonical kinetic term which are denoted by ϕ and $X = -\partial^\mu\phi\partial_\mu\phi/2$, respectively. In this notation, we consider G_2, G_3, G_4, G_5 for ordinary Horndesky theory and F_4, F_5 for beyond Horndeski (bH) theory. These are in the following form

$$\mathcal{L}_2 = G_2, \tag{2}$$
$$\mathcal{L}_3 = -G_3\Box\phi, \tag{3}$$
$$\mathcal{L}_4 = G_4R + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \tag{4}$$
$$\mathcal{L}_5 = G_5G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3], \tag{5}$$
$$\mathcal{L}_4^{bH} = F_4\epsilon^{\mu\nu\rho\sigma}\epsilon_\sigma^{\alpha\beta\gamma}(\nabla_\mu\phi\nabla_\alpha\phi)(\nabla_\nu\nabla_\beta\phi)\nabla_\rho\nabla_\gamma\phi, \tag{6}$$
$$\mathcal{L}_5^{bH} = F_5\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}(\nabla_\mu\phi\nabla_\alpha\phi)(\nabla_\nu\nabla_\beta\phi)(\nabla_\rho\nabla_\gamma\phi)(\nabla_\sigma\nabla_\delta\phi), \tag{7}$$

where R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. For simplicity, in our notation,

$$\Box\phi \equiv g^{\mu\nu}\partial_{\mu\nu}\phi, \qquad (\nabla_\mu\nabla_\nu\phi)^2 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi, \qquad (\nabla_\mu\nabla_\nu\phi)^3 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\rho\phi\nabla_\rho\nabla^\mu\phi,$$

and $G_X = \partial G(X)/\partial X$. Obviously, GR and $f(R)$ gravity are the special limits of the Horndeski gravity which are chosen by $G_2 = G_3 = G_5 = 0, G_4 = 1/2$ and $G_2 = G_3 = G_5 = 0, G_4 = f(R)$, respectively.

In our case, we are interested in to investigate the spherically symmetric and static black hole solutions in shift symmetric Horndeski theories. These black holes are static and asymptotically flat with a static scalar field [17]. So, the static and spherically symmetric ansatz for spacetime and scalar field take the following form respectively

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + sin^2\theta d\varphi^2), \tag{8}$$
$$\phi = \phi(r).$$

Also, we set the G_i functions of the Lagrangian as follows

$$G_2 = \eta X - 2\Lambda,$$
$$G_4 = \zeta + \gamma\sqrt{-X},$$
$$G_3 = G_5 = F_4 = F_5 = 0, \tag{9}$$

where η and γ are dimensionless parameters and Λ is the cosmological constant. The first term of G_2 is a canonical kinetic term and the first term of G_4 is $\zeta = M_{Pl}^2/(16\pi)$ that yields an Einstein-Hilbert term in the action. Finally, the action takes the following form [17]

$$S = \int d^4x\sqrt{-g}\left\{\left[\zeta + \gamma\sqrt{(\partial\phi)^2/2}\right]R - \frac{\eta}{2}(\partial\phi)^2 - 2\Lambda - \frac{\gamma}{\sqrt{2(\partial\phi)^2}}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2]\right\}. \tag{10}$$

The scalar field can be obtained from the metric and scalar field ansatz as [17]

$$\phi' = \pm\frac{\sqrt{2}\gamma}{\eta r^2\sqrt{f}}. \tag{11}$$

For our particular case, the spacetime metric solution takes the following form [17]

$$f(r) = g(r) = 1 - \frac{\mu}{r} - \frac{\gamma^2}{2\zeta\eta r^2} - \frac{\Lambda}{3\zeta}r^2. \tag{12}$$

Explicitly, the solution has the Reissner-Nordström-de sitter (RN+Λ) form. As a consequence of similarity to the RN+Λ form, this solution describes a black hole with mass $\mu/2$

where μ is a free integration constant and electric charge $\sqrt{\frac{-\gamma^2}{2\zeta\eta}}$ for spacetime. The parameters γ and η unavoidably share the same sign. Additionally, this solution has a coordinate singularity (singularities) absorbed in the coordinate transformation(s). Finally, identical to other static solutions, space and time coordinates exchange their roles in the interior of the black hole. Also, $\phi^{'}$ is real for outside of the black hole horizon, $f > 0$, and imaginary for interior of the black hole horizon, $f < 0$. Further, the solution (12) with equation (8), recovers the Reissner-Nordström (RN) metric and the Schwarzschild metric in the limits $\Lambda \to 0$ and $\gamma \to 0$, respectively. Having introduced a particular black hole solution in shift symmetric Horndeski theory, now we study its thermodynamics in the presence of phenomenological quantum gravitational effects encoded in a GUP relation.

3 Thermodynamics of Horndeski Black Holes

To incorporate the GUP effects on the black hole thermodynamics, let us start with the metric to obtain the location of the horizons [68]. The radii of the horizons are determined by the equation $f(r) = 0$. In general, this equation has four roots, which we can classify them as

$$r_1 > r_2 > r_3 > r_4 \tag{13}$$

The lack of cubic term in equation $f(r) = 0$ leads to a negative and unphysical root. So, we have three positive (real) roots that the outermost one is the cosmological horizon while r_2 and r_3 are event horizon and Cauchy horizon. In the current work, we are only interested in the event horizon and Caushy horizon which are given by

$$r_+ = -\frac{1}{2}\sqrt{Y} + \frac{1}{2}\sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}, \tag{14}$$

$$r_- = \frac{1}{2}\sqrt{Y} - \frac{1}{2}\sqrt{\frac{6\zeta}{\Lambda} - Y - \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}, \tag{15}$$

where

$$\begin{aligned} X = \Bigg[& 432\zeta^3\eta^3 - 2592\gamma^2\zeta\eta^2\Lambda - 1944\zeta^2\eta^3\Lambda\mu^2 \\ & + \sqrt{\left[-4\left(36\zeta^2\eta^2 + 72\gamma^2\eta\Lambda\right)^3 + \left(432\zeta^3\eta^3 - 2592\gamma^2\zeta\eta^2\Lambda - 1944\zeta^2\eta^3\Lambda\mu^2\right)^2\right]}\Bigg]^{\frac{1}{3}}, \end{aligned} \tag{16}$$

$$Y = \frac{2\zeta}{\Lambda} - \frac{6 \times 2^{\frac{1}{3}}(\zeta^2\eta^2 + 2\gamma^2\eta\Lambda)}{\eta\Lambda X} - \frac{X}{6 \times 2^{\frac{1}{3}}\eta\Lambda}. \tag{17}$$

In the standard framework, the uncertainty principle can be used to obtain the temperature and entropy of black hole [69]. So, in the same way the GUP is capable to modify the temperature and entropy. The black hole thermodynamics in the presence of the GUP are investigated extensively [60–65]. The uncertainty in the position of an emitted particle in the Hawking effect is given by

$$\Delta x = 2r_+ = -\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}. \tag{18}$$

Also, the uncertainty in the energy of the Hawking particle is

$$\Delta E \approx c\Delta p \approx \frac{\hbar c}{\Delta x} = \hbar c\left[-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}\right]^{-1} . \tag{19}$$

In standard cases, the black hole temperature is $T_{BH} = \frac{\kappa}{2\pi} = T_0$, where κ and T_0 are the horizon surface gravity and the Hawking temperature, respectively. What is the black hole temperature in Horndeski theory? For our special case, considering black hole solution equation (12) yields the standard formula again and there is not any temperature shift (for more details see [24]). So, the Hawking temperature is associated to the black hole event horizon radius by

$$T = \frac{1}{4\pi r_+} = \frac{1}{2\pi\Delta x} . \tag{20}$$

Using the Hawking temperature, the Bekenstein-Hawking entropy related to the black hole mass has the following well known form

$$T = \frac{dE}{dS} = \frac{dM}{dS} . \tag{21}$$

Using equation (18), equation (20) and equation (21), we find

$$S = 2A\left(-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}\right)^{-2} \times \int dM\left[-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}\right] , \tag{22}$$

where $A = 4\pi r_+^2$ is considered as the surface area of the black hole event horizon. Note that the solution (22) recovers $T_{Sch} = 4\pi M^2$ in appropriate limit where T_{Sch} is the entropy of the Schwarzschild black hole in the standard framework. To incorporate the quantum gravity effects on the black hole thermodynamics in this shift symmetric Horndeski theory, we take into account the GUP and develop the study in more details. We consider a generalized uncertainty principle that includes a minimal length, a minimal momentum and a maximal momentum. This GUP has the following form [35–38]

$$\Delta x\Delta p \geq \hbar\Big[1 - \alpha\ell_p(\Delta p) + \alpha^2\ell_p^2(\Delta p)^2 + \beta^2\ell_p^2(\Delta x)^2\Big], \tag{23}$$

where ℓ_p is Planck length, α and β are dimensionless parameters which normally are of the order of unity and depend on the quantum gravity approaches. We note that while α addresses the existence of the ultra-violet (UV) cutoff, β has the origin on the infra-red (IR) sector of the underlying quantum field theory. Solving this relation for Δp gives us the following momentum uncertainty

$$\frac{\Delta p}{\hbar} = \frac{(\alpha\hbar\ell_p + 2r_+)}{2\alpha^2\hbar^2\ell_p^2}\left(1 \pm \sqrt{1 - \frac{4\alpha^2\hbar\ell_p^2(\hbar + 4\beta^2\hbar\ell_p^2 r_+^2)}{(\alpha\hbar\ell_p + 2r_+)^2}}\right) . \tag{24}$$

We can show that this solution has a minimal length, $(\Delta x)_{min} = \alpha\ell_p$, , a minimal momentum, $(\Delta p)_{min} = 2\beta\ell_p$, and a maximal momentum, $(\Delta p)_{max} \simeq \frac{1}{\alpha\ell_p}$ with $c = 1$. Using the series expansion

$$\begin{aligned}\frac{\Delta p}{\hbar} = \frac{1}{\Delta x}\bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3}\\ + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4} + O(\ell_p^5)\bigg) ,\end{aligned} \tag{25}$$

and substituting equation (25) into equation (23), we find the GUP corrected position uncertainty as

$$\Delta x' = \Delta x\Bigg[\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg)^{-1} - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{\alpha^2\ell_p^2\hbar^2}{\Delta x^2}\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg) + \beta^2\ell_p^2\Delta x^2\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg)^{-3} - 2\hbar\alpha\beta^2\ell_p^3\Delta x\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg)^{-2}\Bigg]. \tag{26}$$

So, the modified Hawking temperature for a GUP-corrected Horndeski black hole without charge can be obtained as follows

$$T' = \frac{1}{2\pi\Delta x'} = T\Bigg[\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg)^{-1} - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{\alpha^2\ell_p^2\hbar^2}{\Delta x^2}\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg) + \beta^2\ell_p^2\Delta x^2\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg)^{-3} - 2\hbar\alpha\beta^2\ell_p^3\Delta x\Bigg(1 - \frac{\alpha\hbar\ell_p}{\Delta x} + \frac{(2\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^2}{\Delta x^2} - \frac{\alpha\hbar(4\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^3}{\Delta x^3} + \frac{3\alpha^2\hbar^2(3\alpha^2\hbar^2 + \beta^2\Delta x^4)\ell_p^4}{\Delta x^4}\Bigg)^{-2}\Bigg]^{-1}, \tag{27}$$

where Δx is given by equation (18) and T is the standard Bekenstein-Hawking temperature. Figure 1 shows the evolution of the temperature of the black hole versus Δx. Actually, the final state of the Hawking evaporation of a black hole in the presence of quantum gravitational effect, encoded in the GUP, is a black hole remnant. While a Horndeski black hole in the absence of the GUP corrections evaporates so that its final temperature diverges in the same manner that happens for the Schwarzschild black hole, for the Horndeski black hole modified by the GUP, the temperature increases by evaporation of the black hole. This increment continues up to a maximum temperature and then the temperature reduces to zero for a black hole remnant of essentially Planck size. So, here there is a difference with the Schwarzschild case: As has been shown by Adler et. al. [60], the final state of a GUP-modified Schwarzschild black hole is a remnant with a non-zero temperature. But here the final temperature of the GUP-modified Horndeski black hole is a zero-temperature remnant. Moreover, the behavior of the temperature in terms of the GUP parameters is shown in Figure 2. It seems that one may obtain a maximum temperature by choosing suitable GUP parameters. We note that since black hole evaporation in its final stage towards a stable remnant is essentially a short distance phenomena, the effect of maximal momentum as an IR effect is not so significance in Figure 2.

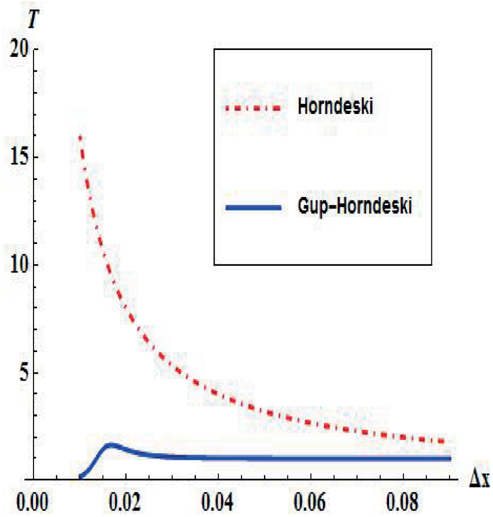

Figure 1: Evolution of the temperature of the black hole versus Δx. The solid line is the GUP corrected Horndesli black hole temperature and the dashed-dot line is the GUP corrected Schwarzschild temperature. The constants are considered to be unity.

Now it is possible to calculate the modified entropy from equation (21) as follows

$$S'(M) = S(M) - 7\ell_p^5 F(M) + O(\ell_p^6), \tag{28}$$

where $F(M)$ is given by

$$F(M) = \int dM \Bigg[2A\Big(\left(\alpha^3\beta^2\hbar^3\right) \left(-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}} \right)^{-2} + \left(3\alpha^5\hbar^5\right) \left(-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}} \right)^{-6} \Big) \Bigg], \tag{29}$$

and the surface area of the black hole's outer horizon is given by

$$A = 4\pi r_+^2 \,. \tag{30}$$

We note that the existence of maximal momentum and minimal momentum leads to extra terms in Hawking temperature and we have both of even and odd powers of Planck length, ℓ_p, in comparison to the results reported in [68]. Furthermore, the $F(M)$ term in the (29) only exists in the presence of the GUP. So, in the absence of GUP, this term vanishes and equation (29) reduces to the Schwarzschild entropy in the standard framework as is expected.

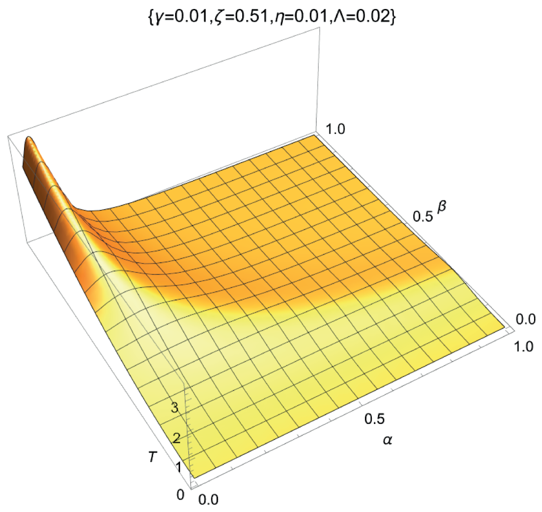

Figure 2: Evolution of the temperature of the black hole versus GUP parameters.

3.1 Heat Capacity

The heat capacity of black hole, in the semiclassical approach, can be obtained by the inverse temperature, $T^{-1} = \beta = \frac{dS}{dM}$. Generally, the heat capacity can be calculated as follows

$$C = \frac{dM}{dT}, \tag{31}$$

$$C = \frac{1}{\pi T^2}\left[\frac{1}{\sqrt{Y}} + \left(\frac{6\zeta}{\Lambda} - Y + \frac{12\zeta M}{\Lambda\sqrt{Y}}\right)^{-\frac{1}{2}}\left(1 + \frac{6\zeta M}{\Lambda Y^{\frac{3}{2}}}\right)\right]^{-1}\frac{1}{W}, \tag{32}$$

where W and Z are given by

$$W = \frac{dY}{dX}Z = \left[\frac{6\times 2^{\frac{1}{3}}\left(\zeta^2\eta^2 + 2\gamma^2\eta\Lambda\right)}{\eta\Lambda X^2} - \frac{1}{6\times 2^{\frac{1}{3}}\eta\Lambda}\right]Z, \tag{33}$$

and

$$Z = \frac{dX}{dM} = \frac{1}{3}X^{-2}\Bigg(15552\zeta^2\eta^3\Lambda M + \frac{1}{2}\Big[X^3 - 432\zeta^3\eta^3 - 592\gamma^2\zeta\eta^2\Lambda - 7774\zeta^2\eta^3\Lambda M\Big]^{-1}\left[2\left(432\zeta^3\eta^3 - 2592\gamma^2\zeta\eta^2\Lambda - 7776\zeta^2\eta^3\Lambda M\right)15552\zeta^2\eta^3\Lambda M\right]\Bigg). \tag{34}$$

If we consider the GUP, we get to

$$C' = C\left[1 - 7\ell_p^5\left(\frac{\alpha^3\beta^2\hbar^3}{(-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}})} + \frac{3\alpha^5\hbar^5}{(-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}})^5}\right)\right]^2 \left[7\ell_p^5\left(\frac{\alpha^3\beta^2\hbar^3}{\left(-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}\right)^2} + \frac{15\alpha^5\hbar^5}{\left(-\sqrt{Y} + \sqrt{\frac{6\zeta}{\Lambda} - Y + \frac{6\zeta\mu}{\Lambda\sqrt{Y}}}\right)^6}\right)\right]^{-1}, \tag{35}$$

where C is the standard Bekenstein-Hawking heat capacity. When α and γ, the GUP parameters tend to zero, equation (35) reduces to equation (32) as would be expected.

The parameter space of the GUP-corrected Horndeski black hole is obviously wider than the parameter space of the Schwarzschild and also standard Horndeski black hole. So, for a GUP-corrected Horndeski black hole, it is essentially possible to find some subspaces of the model parameter space to fulfill the issue of stability. Obviously incorporation of the GUP shifts the point of phase transition in C-T diagram. So, the simple effect of these correction is a shift in the temperature of phase transition. On the other hand, this wider parameter space makes it possible to find more subspaces of the parameter space of the model to fulfill the stability. We note that the heat capacity in this framework has some discontinuities versus temperature that reflect essentially the phase transition. However, due to the wide parameter space of the model it is hard to show these phase transition in a suitable figure. So, we discarded to present a figure here.

3.2 Tunneling Process

In 1974, Stephen Hawking demonstrated [70] that black holes have an emission spectrum of a black body nature, the so called Hawking radiation, and so are not purely black. In 2000, Parikh and Wilczek exhibited [71] a semiclassical method to derive Hawking radiation as a tunneling process from the event horizon of black hole. In this section, we calculate Hawking radiation in Parikh-Wilczek tunneling formalism. For this purpose, the coordinate system should be well-behaved for calculations at the event horizon. So, we define the Painlevé-like coordinate transformation as follows [72–74]

$$dt_R = dt + f^{'}(r)dr, \tag{36}$$

where t_R is the black hole time coordinate. Substituting equation (36) into equation (8) we have

$$\begin{aligned} ds^2 &= -\Delta dt_R^2 + \frac{1}{\Delta}dr^2 + r^2 d\Omega^2 \\ &= -\Delta dt^2 + (-\Delta f^{'2} + \frac{1}{\Delta})dr^2 - 2\Delta f^{'} dr dt + r^2 d\Omega^2 \,, \end{aligned} \tag{37}$$

where for the sake of simplicity we have defined $\Delta = 1 - \frac{\mu}{r} - \frac{\gamma^2}{2\zeta\eta r^2} - \frac{\Lambda}{3\zeta} r^2$. Then, $f^{'}(r)$ satisfies

$$f^{'} = \pm\frac{\sqrt{1-\Delta}}{\Delta}, \tag{38}$$

and the Painlevé line element and the radial geodesics take the following form respectively

$$ds^2 = -\Delta dt^2 + dr^2 \mp 2\sqrt{1-\Delta} dr dt + r^2 d\Omega \,, \tag{39}$$

$$\dot{r} = \frac{dr}{dt} = \pm 1 \mp \sqrt{\frac{\mu}{r} + \frac{\gamma^2}{2\zeta\eta r^2} + \frac{\Lambda}{3\zeta} r^2}. \tag{40}$$

In this process, that occurs near inside the horizon, the particle with positive energy, $\tilde{w}$, tunnels out and escapes the event horizon. Considering the energy conservation, the mass parameter will be replaced with $\mu \to \mu - \tilde{w}$. We can rewrite the new line element and the radial null geodesics which are respectively as

$$ds^2 = -\tilde{\Delta} dt^2 + dr^2 \mp 2\sqrt{1-\tilde{\Delta}} dr dt + r^2 d\Omega \,, \tag{41}$$

and

$$\dot{r} = \frac{dr}{dt} = \pm 1 \mp \sqrt{\frac{(\mu - \tilde{\omega})}{r} + \frac{\gamma^2}{2\zeta\eta r^2} + \frac{\Lambda}{3\zeta} r^2}, \tag{42}$$

where $\tilde{\Delta} = 1 - \frac{(\mu-\tilde{\omega})}{r} - \frac{\gamma^2}{2\zeta\eta r^2} - \frac{\Lambda}{3\zeta} r^2$. To compute the tunneling rate, as a semi-classical procedure, we consider the Wentzel-Kramers-Brillouin (WKB) approximation. The tunneling probability is the imaginary part of the action

$$\Gamma \sim \exp(-2 \operatorname{Im} S). \tag{43}$$

The imaginary part of the particle action across the event horizon, r_+, from initial position, r_{in}, to the final position, r_{out}, is defined as

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} p_r dr = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} d\tilde{p}_r dr, \tag{44}$$

where p_r is the canonical momentum of the outgoing particle. By using the Hamilton's canonical equation

$$\dot{r} = \frac{dH}{dp_r} = \frac{d(\mu - \tilde{\omega})}{dp_r}, \tag{45}$$

and by substituting equation (45) into equation (44) we get

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{(-d\tilde{\omega})dr}{\dot{r}} = Im \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{(-d\tilde{\omega})dr}{1 - \sqrt{\frac{(\mu-\tilde{\omega})}{r} + \frac{\gamma^2}{2\zeta\eta r^2} + \frac{\Lambda}{3\zeta} r^2}}. \tag{46}$$

The commutation relation, from the GUP expression, in the presence of minimal length, minimal momentum and maximal momentum (with $\hbar = 1$) is

$$[r, p_r] = i \left(1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2 + \beta^2 \ell_p^2 r^2\right). \tag{47}$$

In the classical limit we can rewrite this relation between the radial coordinate and the conjugate momentum by poisson bracket

$$\{r, p_r\} = \left(1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2 + \beta^2 \ell_p^2 r^2\right). \tag{48}$$

So, we obtain the deformed Hamiltonian equation as follows

$$\dot{r} = \{r, H\} = \{r, p_r\} \frac{dH}{dr}. \tag{49}$$

Finally, we can rewrite the imaginary part of the action in the presence of GUP as follows

$$\begin{aligned} \operatorname{Im}\ S = {} & \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{\hbar(1 - \alpha \ell_p \tilde{\omega} + \alpha^2 \ell_p^2 \tilde{\omega}^2)}{1 - \sqrt{\frac{(\mu-\tilde{\omega})}{r} + \frac{\gamma^2}{2\zeta\eta r^2} + \frac{\Lambda}{3\zeta} r^2}} (-d\tilde{\omega}) dr \\ & + \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{\hbar(\gamma^2 \ell_p^2 r^2)}{1 - \sqrt{\frac{(\mu-\tilde{\omega})}{r} + \frac{\gamma^2}{2\zeta\eta r^2} + \frac{\Lambda}{3\zeta} r^2}} (-d\tilde{\omega}) dr. \end{aligned} \tag{50}$$

The integral takes the following form up to the second order in l_p

$$\begin{aligned} \operatorname{Im} S \approx \operatorname{Im} \int_{r_{in}}^{r_{out}} & \Bigg[2\pi r - 2\alpha \ell_p \pi \left(\mu r - r^2 + Q^2 + \frac{\Lambda r^4}{3\zeta} \right) + 2\pi \alpha^2 \ell_p^2 \Big[\left(\mu^2 - 2Q^2\right) r - 2\mu r^2 \\ & + 2\mu Q^2 + \frac{2\mu \Lambda r^4}{3\zeta} + \left(1 + \frac{2Q^2 \Lambda}{3\zeta}\right) r^3 + Q^4 \frac{1}{r} + \left(\frac{\Lambda}{3\zeta}\right)^2 r^7 + \frac{2\Lambda r^5}{3\zeta} \Big] + 2\pi \beta^2 \ell_p^2 r^3 \Bigg] dr, \end{aligned} \tag{51}$$

where $Q=\frac{\gamma^2}{2\zeta\eta}$. Therefore, the imaginary part of the action takes the following form

$$\begin{aligned}\operatorname{Im} S \approx & -\pi\left(r_{out}^2-r_{in}^2\right)\left[-1+\alpha\ell_p\mu-\alpha^2\ell_p^2\mu^2+2\alpha^2\ell_p^2Q^2\right]-2\pi\frac{\left(r_{out}^3-r_{in}^3\right)}{3}\left[-\alpha\ell_p+2\alpha^2\ell_p^2\mu\right]\\ & -2\pi\left(r_{out}-r_{in}\right)\left[2\alpha\ell_pQ^2-2\alpha^2\ell_p^2\mu Q^2\right]-2\pi\frac{\left(r_{out}^5-r_{in}^5\right)}{5}\left[\alpha\ell_p\frac{\Lambda}{3\zeta}-2\mu\Lambda\alpha^2\ell_p^2\right]\\ & +2\pi\alpha^2\ell_p^2\frac{\Lambda}{9\zeta}\left(r_{out}^6-r_{in}^6\right)-\pi\frac{\left(r_{out}^4-r_{in}^4\right)}{2}\left[\beta^2\ell_p^2-2\alpha^2\ell_p^2\frac{Q^2\Lambda}{3\zeta}\right]\\ & +2\pi\alpha^2\ell_p^2Q^4\left(\ln r_{out}-\ln r_{in}\right)+2\pi\alpha^2\ell_p^2\left(\frac{\Lambda}{3\zeta}\right)^2\frac{\left(r_{out}^8-r_{in}^8\right)}{8}.\end{aligned} \tag{52}$$

Substituting equation (52) into equation (43), we obtain the tunneling rate at the horizon as follows

$$\begin{aligned}\Gamma \approx \exp\Bigg\{ & 2\pi\left(r_{out}^2-r_{in}^2\right)\left[-1+\alpha\ell_p\mu-\alpha^2\ell_p^2\mu^2+2\alpha^2\ell_p^2Q^2\right]+4\pi\frac{\left(r_{out}^3-r_{in}^3\right)}{3}\left[-\alpha\ell_p+2\alpha^2\ell_p^2\mu\right]\\ & +4\pi\left(r_{out}-r_{in}\right)\left[2\alpha\ell_pQ^2-2\alpha^2\ell_p^2\mu Q^2\right]+4\pi\frac{\left(r_{out}^5-r_{in}^5\right)}{5}\left[\alpha\ell_p\frac{\Lambda}{3\zeta}-2\mu\Lambda\alpha^2\ell_p^2\right]\\ & -4\pi\alpha^2\ell_p^2\frac{\Lambda}{9\zeta}\left(r_{out}^6-r_{in}^6\right)+2\pi\frac{\left(r_{out}^4-r_{in}^4\right)}{2}\left[\beta^2\ell_p^2-2\alpha^2\ell_p^2\frac{Q^2\Lambda}{3\zeta}\right]-4\pi\alpha^2\ell_p^2Q^4\left(\ln r_{out}-\ln r_{in}\right)\\ & -4\pi\alpha^2\ell_p^2\left(\frac{\Lambda}{3\zeta}\right)^2\frac{\left(r_{out}^8-r_{in}^8\right)}{8}\Bigg\}=\exp\left(\Delta S_{BH}\right),\end{aligned} \tag{53}$$

where $\Delta S_{BH}=S_{BH}(\mu-\omega)-S_{BH}(\mu)$ is the difference in Bekenstein-Hawking entropy before and after the particles emission at the event horizon. When $\gamma=0$ and $\Lambda=0$, the result reduces to the Schwarzschild black hole's result [71,72]. Because of the extra terms in comparison to the results of [66], the emission spectrum is not purely thermal.

Finally, we calculate the possible correlation between the emitted particles (modes) that can be obtained by the following relation

$$\chi(E_1+E_2;E_1,E_2)\equiv\ln[\Gamma(E_1+E_2)]-\ln[\Gamma(E_1)\Gamma(E_2)]\,, \tag{54}$$

where $\ln[\Gamma(E_1)]$ and $\ln[\Gamma(E_2)]$ are the emission rates for the first and second emitted particles and $\ln[\Gamma(E_1+E_2)]$ is the emission rate for a single, composed particle with energy $E=E_1+E_2$. The emission rate for the first quanta that carries out the energy E_1 is given by

$$\begin{aligned}\ln[\Gamma(E_1)]\approx & 2\pi r^2\left[-1+\alpha\ell_p(\mu-E_1)-\alpha^2\ell_p^2(\mu-E_1)^2+2\alpha^2\ell_p^2Q^2\right]\\ & +\frac{4\pi r^3}{3}\left[-\alpha\ell_p+2\alpha^2\ell_p^2(\mu-E_1)\right]+4\pi r\left[2\alpha\ell_pQ^2-2\alpha^2\ell_p^2(\mu-E_1)Q^2\right]\\ & +\frac{4\pi r^5}{5}\left[\alpha\ell_p\frac{\Lambda}{3\zeta}-2(\mu-E_1)\Lambda\alpha^2\ell_p^2\right]-4\pi\alpha^2\ell_p^2\frac{\Lambda}{9\zeta}r^6+\frac{2\pi r^4}{2}\left[\beta^2\ell_p^2-2\alpha^2\ell_p^2\frac{Q^2\Lambda}{3\zeta}\right]\\ & -4\pi\alpha^2\ell_p^2Q^4\ln r-\frac{4\pi\alpha^2\ell_p^2r^8}{8}\left(\frac{\Lambda}{3\zeta}\right)^2.\end{aligned} \tag{55}$$

Similarly, the emission rate for the second quanta that carries out the energy E_2 is as follows

$$\begin{aligned}
\ln[\Gamma(E_2)] \approx{}& 2\pi r^2 \left[-1 + \alpha\ell_p((\mu - E_1) - E_2) - \alpha^2\ell_p^2((\mu - E_1) - E_2)^2 + 2\alpha^2\ell_p^2 Q^2\right] \\
&+ \frac{4\pi r^3}{3}\left[-\alpha\ell_p + 2\alpha^2\ell_p^2((\mu - E_1) - E_2)\right] + 4\pi r\left[2\alpha\ell_p Q^2 - 2\alpha^2\ell_p^2((\mu - E_1) - E_2)Q^2\right] \\
&+ \frac{4\pi r^5}{5}\left[\alpha\ell_p\frac{\Lambda}{3\zeta} - 2((\mu - E_1) - E_2)\Lambda\alpha^2\ell_p^2\right] - 4\pi\alpha^2\ell_p^2\frac{\Lambda}{9\zeta}r^6 \\
&+ \frac{2\pi r^4}{2}\left[\beta^2\ell_p^2 - 2\alpha^2\ell_p^2\frac{Q^2\Lambda}{3\zeta}\right] - 4\pi\alpha^2\ell_p^2 Q^4\ln r - \frac{4\pi\alpha^2\ell_p^2 r^8}{8}\left(\frac{\Lambda}{3\zeta}\right)^2 .
\end{aligned} \tag{56}$$

Now, the emission rate for a single quanta that carries out the energy $E_1 + E_2$ is given by

$$\begin{aligned}
\ln[\Gamma(E_1 + E_2)] \approx{}& 2\pi r^2\left[-1 + \alpha\ell_p(\mu - E_1 - E_2) - \alpha^2\ell_p^2(\mu - E_1 - E_2)^2 + 2\alpha^2\ell_p^2 Q^2\right] \\
&+ \frac{4\pi r^3}{3}\left[-\alpha\ell_p + 2\alpha^2\ell_p^2(\mu - E_1 - E_2)\right] + 4\pi r\left[2\alpha\ell_p Q^2 - 2\alpha^2\ell_p^2(\mu - E_1 - E_2)Q^2\right] \\
&+ \frac{4\pi r^5}{5}\left[\alpha\ell_p\frac{\Lambda}{3\zeta} - 2(\mu - E_1 - E_2)\Lambda\alpha^2\ell_p^2\right] - 4\pi\alpha^2\ell_p^2\frac{\Lambda}{9\zeta}r^6 \\
&+ \frac{2\pi r^4}{2}\left[\beta^2\ell_p^2 - 2\alpha^2\ell_p^2\frac{Q^2\Lambda}{3\zeta}\right] - 4\pi\alpha^2\ell_p^2 Q^4\ln r - \frac{4\pi\alpha^2\ell_p^2 r^8}{8}\left(\frac{\Lambda}{3\zeta}\right)^2 .
\end{aligned} \tag{57}$$

The non-zero statistical correlation function can be calculated as

$$\begin{aligned}
\chi(E_1 + E_2; E_1, E_2) \approx{}& (-2\pi r^2 - 4\pi^2 r^4) + \Big(8\pi Q^2 r\alpha - \frac{4}{3}\pi r^3\alpha + \frac{4\pi r^5\alpha\Lambda}{15\zeta} + 2\pi r^2\alpha(\mu - E_1 - E_2) \\
&+ \frac{4\pi^2 r^3(120Q^2\alpha\zeta - 20r^2\alpha\zeta + 4r^4\alpha\Lambda + 30r\alpha\zeta\mu - 30r\alpha\zeta E_1 - 15r\alpha\zeta E_2)}{15\zeta}\Big)\ell_p \\
&+ (-4\ln r\pi Q^4\alpha^2 - \frac{4\pi r^6\alpha^2\Lambda}{9\zeta} - \frac{\pi r^8\alpha^2\Lambda^2}{18\zeta^2} + \pi r^4(\beta^2 - \frac{2Q^2\alpha^2\Lambda}{3\zeta}) \\
&+ 2\pi r^2(-4\ln r\pi Q^4\alpha^2 - \frac{4\pi r^6\alpha^2\Lambda}{9\zeta} - \frac{\pi r^8\alpha^2\Lambda^2}{18\zeta^2} + \pi r^4(\beta^2 - \frac{2Q^2\alpha^2\Lambda}{3\zeta}) \\
&+ 2\pi r^2(2Q^2\alpha^2 - \alpha^2(\mu - E_1)^2) - \frac{8}{5}\pi r^5\alpha^2\Lambda(\mu - E_1) + \frac{8}{3}\pi r^3\alpha^2(\mu - E_1) \\
&+ 8\pi Q^2 r\alpha^2(-\mu + E_1))\Big(8\pi Q^2 r\alpha - \frac{4}{3}\pi r^3\alpha + \frac{4\pi r^5\alpha\Lambda}{15\zeta} + 2\pi r^2\alpha(\mu - E_1)\Big) \\
&\Big(8\pi Q^2 r\alpha - \frac{4}{3}\pi r^3\alpha + \frac{4\pi r^5\alpha\Lambda}{15\zeta} + 2\pi r^2\alpha(\mu - E_1 - E_2)\Big) - \frac{8}{5}\pi r^5\alpha^2\Lambda(\mu - E_1 - E_2) \\
&+ \frac{8}{3}\pi r^3\alpha^2(\mu - E_1 - E_2) + 8\pi Q^2 r\alpha^2(-\mu + E_1 + E_2) + 2\pi r^2\alpha^2(2Q^2 - (\mu - E_1 - E_2)^2) \\
&+ 2\pi r^2\Big(- 4\ln r\pi Q^4\alpha^2 - \frac{4\pi r^6\alpha^2\Lambda}{9\zeta} - \frac{\pi r^8\alpha^2\Lambda^2}{18\zeta^2} + \pi r^4(\beta^2 - \frac{2Q^2\alpha^2\Lambda}{3\zeta} \\
&- \frac{8}{5}\pi r^5\alpha^2\Lambda(\mu - E_1 - E_2) + \frac{8}{3}\pi r^3\alpha^2(\mu - E_1 - E_2) + 8\pi Q^2 r\alpha^2(-\mu + E_1 + E_2) \\
&+ 2\pi r^2\alpha^2(2Q^2 - (\mu - E_1 - E_2)^2))\Big)\ell_p^2 + O(\ell_p^3).
\end{aligned} \tag{58}$$

Obviously, the statistical correlation function is not zero. So, black hole radiation is not purely thermal. Also, existence of the non-zero correlation means that information can come out during the evaporation process. Since these correlations can store some sort of information, so these correlations are capable to address at least part of the lost information in essence.

It is important to point that these correlations are essentially quantum mechanical in nature and usually are spatial/temporal correlations. But here, the spatial correlations of these successive emitted modes (particles) is considered. These correlations are correlation between two successive particles emitted via Hawking radiation and it is obvious, it is a function of the energies of the emitted particles, the radial coordinate, the GUP parameters and also the Horndeski gravitational theory.

4 Discussion

While the issue of black hole thermodynamics in the framework of the generalized/extended uncertainty relations has been studied widely in literature, the issue of Horndeski black holes' thermodynamics in the framework of GUP/EUP has been overlooked in literature. On the other hand, Horndeski theory provides the most general framework for scalar-tensor theories of gravity. For the reasons, we have stated in Introduction, in this paper we have focused on the thermodynamics of shift symmetric Horndeski black hole solutions in the framework of phenomenological quantum gravity corrections encoded in a class of generalized/extended uncertainty relation. We obtained in details the temperature and then the heat capacity of such a black hole that recovers the standard Schwarzschild, Reissner-Nordström or Reissner-Nordström-de sitter solutions in the appropriate limits. While a Horndeski black hole in the absence of the GUP corrections evaporates so that its final temperature diverges in the same manner that happens for the Schwarzschild black hole, for the Horndeski black hole modified by the GUP, the temperature increases by evaporation of the black hole. This increment continues up to a maximum temperature and then the temperature reduces to zero for a black hole remnant of essentially Planck size. Then, the issue of Hawking radiation as a semi-classical tunneling from the event horizon has been studied in details. For this purpose, the imaginary part of the classical action has been calculated within the WKB approximation. The issue of possible correlations between the emitted modes (particles) has been treated carefully and it is shown that these correlations are not vanishing, leading to the conclusion that part of the lost information may be stored in these quantum gravitational correlations. It is important to note that we focused mainly on the near, "event" horizon calculations. There is in fact some correlations between the various horizons of these multi-horizon geometry [75] and these correlations should be taken into account in a more realistic and concrete study. We leave this issue for our forthcoming study.

Authors' Contributions

All authors have the same contribution.

Data Availability

No data available.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or nonprofit sectors.

References

[1] Berti, E. et al., 2015, Class. Quant. Grav., 32, 243001.

[2] Clifford, M. W. 2014, Living Rev. Rel., 17, 4.

[3] Riess, A. G., & et al. 1998, ApJ, 116, 1009.

[4] Perlmutter, S., & et al. 1999, ApJ, 517, 565.

[5] Sofue, Y., & Rubin, V. 2001, Annu. Rev. Astron. Astrophys., 39, 137.

[6] Clifton, T., Ferreira, P. G., Padilla, A., & Skordis, C. 2012, Phys. Rept. 513, 1.

[7] Papantonopoulos, E. 2015, Lect. Notes Phys., 892.

[8] Horndeski, G. W. 1974, Int. J. Theoret. Phys., 10, 363.

[9] Saridakis, E. N., & Sushkov, S. V. 2010, Phys. Rev. D, 81, 083510.

[10] Charmousis, C., Copeland, E. J., Padilla, A., & Saffin, P. M., 2012, Phys. Rev. Lett., 108, 051101.

[11] Zumalacárregui, M., & Garcá-Bellido, J. 2014, Phys. Rev. D, 89, 064046.

[12] Gleyzes, J., Langlois, D., Piazza, F., & Vernizzi, F. 2015, Phys. Rev. Lett., 114, 211101.

[13] Langlois, D., & Noui, K. 2016, JCAP, 02, 034.

[14] Kobayashi, T. 2019, Rept. Prog. Phys., 82, 086901.

[15] Rinaldi, M. 2012, Phys. Rev. D, 86, 084048.

[16] Babichev, E., Charmousis, C., & Lehébel, A. 2016, Class. Quantum Grav., 33, 154002.

[17] Babichev, E., Charmousis, C., & Lehébel, A. 2017, JCAP, 04, 027.

[18] Anabalon, A., Cisterna, A., & Oliva, J. 2014, Phys. Rev. D, 89, 084050.

[19] Cisterna, A., & Erices, C. 2014, Phys. Rev. D, 89, 084038.

[20] Badía, J., & Eiroa, E. F. 2017, Eur. Phys. J. C, 77, 779.

[21] Kumar, J., Islam, S. U., & Ghosh, S. G. 2022, Eur. Phys. J. C, 82, 443.

[22] Afrin, M., & Ghosh, S. G. 2022, ApJ, 932, 51.

[23] Miao, Y. G., Xu, Z. M. 2016, Eur. Phys. J. C, 76, 638.

[24] Hajian, K., Liberati, S., Sheikh-Jabbari, M. M., & Vahidinia, M. H. 2021, Phys. Lett. B, 812, 136002.

[25] Walia, R. K., Maharaj, S. D., & Ghosh, S. G. 2022, Eur. Phys. J. C, 82, 547.

[26] Salahshoor, Kh., & Nozari, K. 2018, Eur. Phys. J. C, 78, 486.

[27] Veneziano, G. 1986, Europhys. Lett., 2, 199.

[28] Amati, D., Cialfaloni, M., & Veneziano, G. 1989, Phys. Lett. B, 216, 41.

[29] Gross, D. J., & Mende, P. F. 1987, Phys. Lett. B, 197, 129.

[30] Konishi, K., Paffuti, G., & Provero, P. 1990, Phys. Lett. B, 234, 276.

[31] Garay, L. J. 1995, Int. J. Mod. Phys. A, 10, 145.

[32] Capozziello, S., Lambiase, G., & Scarpetta, G. 2000, Int. J. Theor. Phys., 39, 15.

[33] Maggiore, M. 1993, Phys. Lett. B, 304, 65.

[34] Scardigli, F. 1999, Phys. Lett. B, 452, 39.

[35] Kempf, A. 1994, J. Math. Phys., 35, 4483.

[36] Kempf, A., Mangano, G., & Mann, R. B. 1995, Phys. Rev. D, 52, 1108.

[37] Hinrichsen, H., & Kempf, A. 1996, J. Math. Phys., 37, 2121.

[38] Kempf, A. 1997, J. Math. Phys., 38, 1347.

[39] Hossenfelder, S. 2012, Class. Quant. Grav., 29, 115011.

[40] Nozari, K., & Etemadi, A. 2012, Phys. Rev. D, 85, 104029.

[41] Hossenfelder, S. 2013, Living Rev. Relativ., 16, 2.

[42] Tawfik, A., & Diab, A. 2014, Int. J. Mod. Phys. D, 23, 1430025.

[43] Nozari, K., & Azizi, T. 2006, Gen. Rel. Grav., 38, 735.

[44] Harbach, U., & Hossenfelder, S. 2006, Phys. Lett. B, 632, 379.

[45] Das, S., & Vagenas, E. C. 2008, Phys. Rev. Lett., 101, 221301.

[46] Basilakos, S., Das, S., & Vagenas, E. C. 2010, JCAP, 1009, 027.

[47] Ali, A. F. 2011, Class. Quant. Grav., 28, 065013.

[48] Amelino-Camelia, G. 2000, Int. J. Mod. Phys. D, 11, 35.

[49] Kowalski-Glikman, J. 2005, Lect. Notes Phys., 669, 131.

[50] Amelino-Camelia, G., Kowalski-Glikman, J., Mandanici, G., & Procaccini, A. 2005, Int. J. Mod. Phys. A, 20, 6007.

[51] Magueijo, J., & Smolin, L. 2002, Phys. Rev. Lett., 88, 190403.

[52] Magueijo, J., & Smolin, L. 2003, Phys. Rev. Lett. D, 67, 044017.

[53] Cortes, J. L., & Gamboa, J. 2005, Phys. Rev. D, 71, 065015.

[54] Ali, A. F., Das, S., & Vagenas, E. C. 2009, Phys. Lett. B, 678, 497.

[55] Ali, A. F., Das, S., & Vagenas, E. C. 2011, Phys. Rev. D, 84, 044013.

[56] Nozari, K., & et al. 2019, Eur. Phys. J. C, 79, 465.

[57] Nozari, K., Gorji, M. A., Hosseinzadeh, V., & Vakili, B. 2016, Class. Quant. Gravit., 33, 025009.

[58] Adler, R. J., & Santiago, D. I. 1999, Mod. Phys. Lett. A, 14, 1371.

[59] Nozari, K. 2007, Astropart. Phys., 27, 169.

[60] Adler, R. J., Chen, P., & Santiago, D. I. 2001, Gen. Rel. Grav., 33, 2101.

[61] Amelino-Camelia, G., Arzano, M., Ling, Y., & Mandanici, G. 2006, Class. Quant. Grav., 23, 2585.

[62] Nozari, K., & Sefiedgar, A. S. 2006, Phys. Lett. B, 635, 156.

[63] Kim, W., Son, E. J., & Yoon, M. 2008, JHEP, 0801, 035.

[64] Banerjee, R., & Ghosh, S. 2010, Phys. Lett. B, 688, 224.

[65] Myung, Y. S., Kim, Y. W., & Park, Y. J. 2007, Phys. Lett. B, 645, 393.

[66] Nozari, K., & Saghafi, S. 2012, JHEP, 11 005.

[67] Vagnozzi, S., & et. al. 2023, Class. Quantum Grav., 40, 165007.

[68] Said, J. L., & Adami, K. Z. 2011, Phy. Rev. D, 83, 043008.

[69] Ohanian, H., & Ruffini, R. 1994, Gravitation and Spacetime, 2nd edition (W. W. Norton), 481.

[70] Hawking, S. W. 1974, Nature, 248, 30.

[71] Parikh, M. K., & Wilczek, F. 2000, Phys. Rev. Lett., 85, 5042.

[72] Medved, A. J. M. 2002, Phys. Rev. D, 66, 124009.

[73] Miao, Y. G., Xue, Z., & Zhang, S. J. 2011, Europhys. Lett., 96, 1000.

[74] Qing-Quan, J., Shu-Zheng Y., & Hui-Ling, L. 2006, Commun. Theor. Phys., 45, 457.

[75] Shankaranarayanan, S. 2003, Phys. Rev. D, 67, 084026.