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Thermodynamic Phase Transition and Joule Thomson Adiabatic Expansion for dS/AdS Bardeen Black Holes with Consistent 4D Gauss-Bonnet Gravity

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Abstract. Instead of the work [1] given by Glaven and Lin in which in according to the Lovelock theorem it is not applicable for all types of 4D curved spacetimes of Einstein Gauss Bonnet gravity, authors of the work [2] applied breakdown of diffeomorphism property to present a consistent Einstein-Gauss-Bonnet gravity theory in 4D. In this work we use the latter model by adding an Ayon-Beato-Garcia type of nonlinear electromagnetic field Lagrangian density to study effects of GB coupling constant on the thermodynamic phase transition and Joule-Thomson adiabatic expansion of a 4D de Sitter/Anti de Sitter Gauss-Bonnet-Bardeen black hole. In fact, we will see importance of parameters of this black hole namely the magnetic charge and the Gauss Bonnet coupling constant parameter on its heating-cooling phase transition. Physical importance of this type of black holes is non-singular property which they have and are applicable to study black hole structure of center of galaxies.

Keywords: Non-singular black holes, Nonlinear Maxwell fields, Thermodynamic physic transition, Joule Thomson effect, Magnetic charge

1 Introduction

In fact, Claven and Lin at a first time at 2020 presented a general covariant modified theory of gravity in $D = 4$ spacetime dimensions which propagates only the massless graviton and bypasses the Lovelock‘s theorem. Their theory is formulated in $D > 4$ dimensions and its action consists of the Einstein-Hilbert term with a cosmological constant, and the Gauss-Bonnet term multiplied by a factor $(D-4)^{-1}$. The four-dimensional theory is defined as the limit, $D \to 4$. In this singular limit the Gauss-Bonnet invariant gives rise to non-trivial contributions to gravitational dynamics, while preserving the number of graviton degrees of freedom and being free from Ostrogradsky instability. Some appealing new predictions of this theory is reported, including the corrections to the dispersion relation of cosmological tensor and scalar modes, singularity resolution for spherically symmetric solutions in the paper [1]. After some months of release of this work, Aoki et al showed some weakness and failure of this work as follows: However, an intriguing idea of [1] is to multiply the GB term by the factor $(D-4)^{-1}$ before taking the limit. It was shown that, at the level of equations of motion under a concrete ansatz of the metric, the divergent factor $(D-4)^{-1}$ is canceled by the vanishing GB contributions yielding finite nontrivial effects. Despite the singular limit, it was conjectured that the $D \to 4$ limit should have only two dofs (dynamical degrees of freedom), based on the fact that the number of dofs of the D-dimensional EGB gravity is $D(D-3)/2$. The original suggestion of the $D \to 4$ EGB gravity is in explicit contradiction with the common knowledge. To resolve the problem some researchers started with a direct product D-dimensional spacetime and then took the limit $D \to 4$[3–8]. They found well-defined theories which belong to a class of Horndeski theory [5] but with $2+1$ dofs, which in general is not sufficient to resolve the problem in 4-dimensions. Because the scalar-tensor description lacks the quadratic kinetic term of the scalar field and thus, suffers from the infinite strong coupling problem in general (see [4,9,10]). In short, by looking at the works [11–14] one infers that there is no manifestly covariant novel $D \to 4$ EGB gravity with only two dofs, in agreement with the Lovelock theorem. Even if one adopts the scalar-tensor description, the spacetimes provided by [1] cannot be realized by a 4-dimensional theory in a consistent manner. According to the Lovelock theorem, if there indeed exists a novel 4-dimensional theory with two dofs, the only possibility is that the system cannot be described in a covariant manner. In other words, this problem can be resolved just by breakdown of 4-dimensional diffeomorphism invariance done by Akoi et al in ref [2]. Hence, we use their model in this work and define it in the next section.

From a theoretical point view, we know black holes are made from metric solutions of the Einstein's field equation with no temperature and so they are not supposed to show any thermodynamic behavior. For the first time, Hawking presented an important theorem where the event horizon of the black holes should never be decreased because all objects are absorbed by them [15]. This is called now as the Hawking's area theorem. After this presentation, Bekenstein suggested that for the black hole should be assigned an entropy appropriate to the area of its horizon[16]. In analogy with the thermodynamics rules of the ordinary systems, four laws proposed for the black holes thermodynamics. But by considering this analogy, there was appeared a problem for thermodynamics of the black holes as follows. Actually, the first law of the thermodynamics of the black holes lacks the pressure and volume components. Because there is not a clear concept for thermodynamic volume and pressure of a black hole. The first idea to solve the pressure problem led to the consideration of a negative cosmological constant [17–21] which it is called conjugate variable for the thermodynamic volume. There are done a lot of research where the pressure-volume (PV) criticality of thermodynamics of AdS black holes mimics thermodynamic behavior of the well known Van der Waals ordinary gases [22–39]. Recently Glavan and Lin released a paper [40] in which an alternative generally covariant gravity theory is defined which in $4D$ curved spacetimes, can propagates just massless gravitons by bypassing the Lovelock's theorem. This alternative higher order derivatives gravity theory has two correction terms called as Gauss-Bonnet topological invariant and cosmological constant respectively. In 4D curved spacetimes the Gauss-Bonnet coupling constant parameter diverges to an infinite value. In this singular limit, the Gauss-Bonnet topological invariant term gives rise nontrivial contributions to the gravitational dynamics, while preserving the number of degrees of freedom of graviton and being free from Ostrogradsky instability. They reported some appealing corrections to the dispersion relation of cosmological tensor-scalar modes in the cosmological spacetimes and also singularity resolution in the spherically symmetric spacetimes. As an spherically symmetric static black hole metric solutions of this model, authors of the work [41] obtained a Bardeen type of the black hole solutions and generated its thermodynamic variables via the horizon calculation. They obtained a critical location for the black hole horizon where the corresponding Hawking temperature raises to a maximum value for which a second-order phase transition happens because the heat capacity diverges to infinity. Existence of these appealing thermodynamic behavior encourages us to study Joule-Thomson adiabatic free expansion phenomena for this black hole given in the AdS background. If we want to describe this expansion briefly, this is down as follows. In fact this expansion happens when a gas is allowed to move from a high pressure region to a low pressure one without to change its enthalpy. As it is established above, the black hole mass would be taken as enthalpy in an extended thermodynamic phase space, so during the Joule-Thomson expansion phenomena the mass remains constant (isentropic process). In presence of this expansion, the black hole usually could reach to one of the two heating or cooling phases finally. After the pioneer works about the black hole Joule-Thomson expansion given by [42–44], many other researchers investigated this phenomena for several black holes interacting with many types of the mater fields [45–60]. To study this phenomena one can usually investigate the inversion curves which mimics behavior of the van der Waals fluid. In general, there are several AdS black holes which behave as different with respect to the van der Waals fluid and they do not mimic completely the inversion curves (see for instance [41,42]). In this work, we want to examine the possibility of the emergence of the expansion phenomenon of Joule-Thomson for dS/AdS Gauss- Bonnet-Bardeen black hole. In fact, we will see importance of parameters of this black hole namely the magnetic charge and the GB coupling constant in treating its heating-cooling phase transition. The paper is organized as follows.

As in the beginning of the introduction, we briefly mentioned the importance of the gravity model used in this article, we define the consistent 4D GB gravity model given by [2] in the second section. Then, as an application of the model we consider matter source of the system to be nonlinear electromagnetic field Lagrangian density of a Bardeen black hole. In the third section, we derive metric field equations of dS/AdS Gauss-Bonnet-Bardeen black hole and solve them by using some physical arguments. In fourth section, we calculate the horizon equation, the Hawking temperature, the equation of state and the Joule-Thomson coefficient for different regimes of the 4D GB dS/AdS black hole mass density function. Then, by plotting P-V diagrams at constant temperature and also the isenthalpic T-P curves we study possibility of the phase transitions and JT expansion of the black hole under consideration. Fifth section is devoted to conclusion and outlook of the work.

2 Consistent 4D EGB gravity

According to the work [2], we define consistent EGB gravity in $D \to 4$ limit with the first term of the Lagrangian density in the following action functional in which the second term of the lagrangian density $\mathcal{L}_{matter}$ denotes the source part

$$I = \frac{1}{16\pi G}\int dt d^3x N\sqrt{\gamma}(\mathcal{L}^{4D}_{EGB} + \mathcal{L}_{matter}), \tag{1}$$

where

$$\mathcal{L}^{4D}_{\text{EGB}} = 2R - 2\Lambda - \mathcal{M} \tag{2}$$
$$+\frac{\tilde{\alpha}}{2}[8R^2 - 4R\mathcal{M} - \mathcal{M}^2 - \frac{8}{3}(8R_{ij}R^{ij} - 4R_{ij}\mathcal{M}^{ij} - \mathcal{M}_{ij}\mathcal{M}^{ij})],$$

and G is the Newton gravitational coupling constant. R and R_{ij} are the Ricci scalar and the Ricci tensor of the spatial 3-metric γ_{ij}, respectively. In the definition (2), we have

$$\mathcal{M}_{ij} = R_{ij} + \mathcal{K}^k_k\mathcal{K}_{ij} - \mathcal{K}_{ik}\mathcal{K}^k_j, \quad \mathcal{M} = \mathcal{M}^i_i \tag{3}$$

with

$$\mathcal{K}_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - 2D_iN_j - 2D_jN_i - \gamma_{ij}D_kD^k\lambda_{GF}). \tag{4}$$

Here, dot $\cdot$ denotes derivative with respect to the time t and all the effects of the constraint stemming from the $gauge-fixing$ (GF) are now encoded in Lagrange multiplier λ_{GF}. D_i is spatial covariant derivative and re-scaled regular EGB coupling constant $\tilde{\alpha}$ is defined versus the irregular GB coupling constant α_{GB} such that $\tilde{\alpha} = (D-4)\alpha_{GB}$ which in the limit of $D \to 4$ become finite. The above EGB gravity action functional satisfies the following gauge condition for all spherically symmetric and cosmological backgrounds (see [2] and [61]).

$$\sqrt{\gamma} D_k D^k (\pi^{ij}\gamma_{ij}/\sqrt{\gamma}) \approx 0. \tag{5}$$

In fact, the above EGB action functional is generated from ADM decomposition of the 4D background metric as $1+3$ dimensions such that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{6}$$

where N, N_i, γ_{ij} are the lapse function, the shift vector, and the spatial metric respectively. γ factor in the action functional (1) is absolute value of determinant of the spatial 3-metric γ_{ij}. This ADM decomposition is done on the background metric to remove divergent boundary term of the higher order metric derivatives in the GB term of the action functional (1) in general 4 dimensional form [2]. First term in the theory defined by (1) has the time re-parametrization symmetry $t \to t = t(t')$. We now set the matter source I_{matter} to be action of a nonlinear electromagnetic antisymmetric Maxwell field $F_{\mu\nu}$ with Ayon-Beato-Garcia form of the Lagrangian density as follows,

$$\mathcal{L}_{matter} = \mathcal{L}_{ABG}(F) = \frac{12M_{ADM}}{Q^3}\left(\frac{\sqrt{2Q^2F}}{1+\sqrt{2Q^2F}}\right)^{\frac{5}{2}}, \tag{7}$$

where $F = F_{\mu\nu}F^{\mu\nu}$, M_{ADM} is ADM mass of the black hole and Q is Bardeen magnetic charge (see equation (8) in [62]).

3 4D dS/AdS GB Bardeen Black Holes

By comparing the line element (6) with general form of a spherically symmetric static 4D metric field,

$$ds^2 = -e^{2A(r)}\left(1 - \frac{2M(r)}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M(r)}{r}} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{8}$$

we infer that the lapse function N, and the shift vector N_i and the spatial metric components γ_{ij} and gauge fixing lagrange multiplier λ_{GF} should be r dependent so that we can write

$$N = e^{A(r)}\sqrt{1-\frac{2M(r)}{r}}, \quad N_{r,\theta,\varphi} = 0, \tag{9}$$

$$\gamma_{rr} = \frac{1}{1-\frac{2M(r)}{r}} \quad \gamma_{\theta\theta} = r^2, \quad \gamma_{\varphi\varphi} = r^2\sin^2\theta, \quad \lambda_{GF} = \lambda_{GF}(r).$$

By substituting (8), into (2) we obtain

$$\mathcal{L}^{4D}_{EGB} = R(\gamma_{ij}) - 2\Lambda + 12q^2 + \frac{\tilde{\alpha}}{2}\left[3R^2(\gamma) + \frac{88}{3}q^2R(\gamma) - 272q^4 - 8R_{ij}(\gamma)R^{ij}(\gamma)\right], \tag{10}$$

in which

$$q(r) = \frac{e^{-A(r)}}{r^2}\Big[r^2\lambda'_{GF}(r)\Big]', \qquad \mathcal{K}_{ij} = -q\gamma_{ij}, \tag{11}$$

$$R(\gamma) = -\frac{4M'}{r^2}, \qquad R_{ij}(\gamma)R^{ij}(\gamma) = \frac{6}{r^6}(M - rM')^2, \tag{12}$$

and "$'$" denotes derivative with respect to r. From the Maxwell equations, we can prove that the magnetic field has the form [62]

$$F_{\theta\varphi}(\theta) = Q\sin\theta, \tag{13}$$

which for the metric equation (8) the EM lagrangian density become

$$F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{Q^2}{2r^4}, \tag{14}$$

for which (7) leads to the following form [62].

$$\mathcal{L}_{Bardeen}(r) = \frac{12M_{ADM}}{Q^3}\left(\frac{Q^2}{r^2+Q^2}\right)^{\frac{5}{2}}. \tag{15}$$

By adding (15) and by substituting (10) and (12) and by integrating the action functional (1) on the 2-sphere $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$, we obtain

$$I = \frac{1}{4G}\int dt\int drr^2e^{A(r)}\Big\{-\frac{4M'}{r^2} - 2\Lambda + 12q^2 + \frac{12M_{ADM}}{Q^3}\left(\frac{Q^2}{r^2+Q^2}\right)^{\frac{5}{2}} -\tilde{\alpha}\Big[\frac{176M'q^2}{3r^2} + 136q^4 + \frac{24M^2}{r^6} - \frac{48MM'}{r^5}\Big]\Big\}. \tag{16}$$

Euler-Lagrange equation for q reads

$$q\Big[12 - \tilde{\alpha}\Big(\frac{176M'}{3r^2} + 136q^2\Big)\Big] = 0, \tag{17}$$

which has two different solutions as

$$q_1 = 0, \quad q_2 = \frac{\pm 1}{\sqrt{136}}\sqrt{\frac{12}{\tilde{\alpha}} - \frac{176M'}{3r^2}}. \tag{18}$$

By substituting these two different gauge fixing conditions into the equation (11) and the action functional (16), we obtain

$$\lambda^{(1)}_{GF}(r) \sim \frac{1}{r}, \tag{19}$$

$$\lambda^{(2)}_{GF}(r) = \int^r \frac{dr'}{r'^2}\int^{r'} r''^2 q_2(r'')e^{A(r'')}dr'', \tag{20}$$

and

$$I_1 = I_2 = \frac{1}{4G}\int dt\int drr^2e^{A(r)}\Big\{-\frac{4M'}{r^2} - 2\Lambda + \frac{12M_{ADM}}{Q^3}\left(\frac{Q^2}{r^2+Q^2}\right)^{\frac{5}{2}} +\tilde{\alpha}\Big[-\frac{24M^2}{r^6} + \frac{48MM'}{r^5}\Big]\Big\}. \tag{21}$$

This shows that two different gauge fixing conditions $q_{1,2}$ reach to similar action functional $I_1 = I_2$ and therefore similar metric solutions. The Euler-Lagrange equations for the function $A(r)$ and the mass distribution function $M(r)$ reduce to the following relations, respectively

$$\frac{4M'}{r^2} = \frac{\frac{12M_{ADM}}{Q^3}\left(\frac{Q^2}{r^2+Q^2}\right)^{\frac{5}{2}} - 2\Lambda - \frac{24\tilde{\alpha}M^2(r)}{r^6}}{1-\frac{12\tilde{\alpha}M(r)}{r^3}} \tag{22}$$

and

$$A'(r) = \frac{\frac{-24\tilde{\alpha}M(r)}{r^4}}{1-\frac{12\tilde{\alpha}M(r)}{r^3}}. \tag{23}$$

Now, we are in the position to solve the above nonlinear differential equations. Here, we try to obtain analytical solutions of the metric equations which are applicable in studying the black hole thermodynamic. To do so, we pay attention to the equation (22) which up to the Gauss-Bonnet term $\tilde{\alpha} = 0$ reads the following solution

$$M_0(r) = \frac{M_{ADM}r^3}{(r^2+Q^2)^{\frac{3}{2}}} - \frac{\Lambda r^3}{6}, \tag{24}$$

for which

$$A_0 = constant = 0. \tag{25}$$

By substituting $\Lambda = 0$ into the above zero order solution of the mass function we obtain $\lim_{r\to\infty} M_0(r) = M_{ADM}$ and the metric equation (8) reads to the original form of the Bardeen black hole. By applying (24) one can show that the metric field reaches to a vacuum (anti) de Sitter metric solution asymptotically at the center of the black hole and Schwarzschild (anti) de Sitter form of a black hole asymptotically at far from its central regions, such that

$$\lim_{r\ll|Q|}\left(1-\frac{2M_0(r)}{r}\right) \sim \left(1-\frac{\bar{\Lambda}}{3}r^2\right), \quad \bar{\Lambda} = \frac{6M_{ADM}}{Q^3} - \Lambda, \tag{26}$$

and

$$\lim_{r\gg|Q|}\left(1-\frac{2M_0(r)}{r}\right) \sim \left(1-\frac{2M_{ADM}}{r} + \frac{\Lambda}{3}r^2\right). \tag{27}$$

By looking at the asymptotic solution (26), one can infer that the quantity $\frac{6M_{ADM}}{Q^3}$ behaves as alternative cosmological constant to make as nonsingular the central region of the black hole.

However, we try to solve the equation (22) to obtain an analytic solution for the mass function $M(r)$ as follows. According to the zero order solution (24), it is easy to write the equation (22) as follows.

$$\frac{4M'}{r^2} = \frac{\frac{4M_0'}{r^2} - \frac{24\tilde{\alpha}M^2}{r^6}}{1-\frac{12\tilde{\alpha}M}{r^3}} \tag{28}$$

which is equivalent to

$$r^2\frac{d}{dr}(M-M_0) = \frac{6\tilde{\alpha}[r(M^2)' - M^2]}{r^2} = 6\tilde{\alpha}\frac{d}{dr}\left(\frac{M^2}{r}\right). \tag{29}$$

This equation is a nonlinear differential equation for mass function $M(r)$ and its nonlinearity is generated from the GB parameter $\tilde{\alpha}$. To study thermodynamics of this black hole we need an analytic solution, which for nonlinear differential equations it is possible to obtain via perturbation series method for small $\tilde{\alpha}$. To do so we define

$$M(r;\tilde{\alpha}) = M_0(r) + \tilde{\alpha}M_1(r) + \tilde{\alpha}^2 M_2(r) + O(3) \tag{30}$$

in which we defined

$$M_0(r) = M(r;0), \quad M_1(r) = \frac{\partial M}{\partial \tilde{\alpha}}\Big|_{\tilde{\alpha}=0}, \quad M_2(r) = \frac{\partial^2 M}{\partial \tilde{\alpha}^2}\Big|_{\tilde{\alpha}=0} \tag{31}$$

and so on. By substituting the above series expansion into the equation (29) and by keeping coefficients of the GB parameter as different linear differential equations and solving them step by step, we can obtain for up to third order term

$$M_1(r) = \int \frac{6}{r^2} d\left(\frac{M_0^2(r)}{r}\right) = \frac{6M_0^2(r)}{r^3} + 12\int \frac{M_0^2(r)}{r^4}dr, \tag{32}$$

and

$$M_2(r) = \int \frac{12}{r^2} d\left(\frac{M_0 M_1}{r}\right) = \frac{12 M_0 M_1}{r^3} + \int \frac{24 M_0 M_1}{r^3}dr. \tag{33}$$

For small values of the GB parameter $\tilde{\alpha}$, the first order solution of the above mass function is enough to seek for how the GB parameter can affect on the thermodynamic properties of this kind of black holes. Linear order solution of the mass function reads

$$M(r) \approx M_0(r) + \tilde{\alpha}M_1(r), \tag{34}$$

for which $M_0(r)$ should be substituted via the equation (24) and $M_1(r)$ is calculated trivially as follows

$$M_1(r) = \frac{6M_0^2(r)}{r^3} + \frac{\Lambda^2 r^3}{9} + \frac{4\Lambda M_{ADM} r}{\sqrt{r^2+Q^2}} + \tag{35}$$
$$\frac{3M_{ADM}^2}{2Q^3}\tan^{-1}\left(\frac{r}{Q}\right) - 4\Lambda M_{ADM}\sinh^{-1}\left(\frac{r}{Q}\right) + \frac{3M_{ADM}^2 r}{2Q^2(r^2+Q^2)} - \frac{3M_{ADM}^2 r}{(r^2+Q^2)^2}$$

for which the equation (23) reads

$$A(r) = 4\tilde{\alpha}\left\{\Lambda \ln\left(\frac{r}{Q}\right) - \frac{6M_{ADM}}{Q^2\sqrt{Q^2+r^2}} + \frac{6M_{ADM}}{Q^3}\tanh^{-1}\left(\frac{Q}{\sqrt{Q^2+r^2}}\right)\right\}. \tag{36}$$

For this obtained solution, the metric components are

$$g^{rr}(r) = 1 - \frac{2(M_0(r) + \tilde{\alpha}M_1(r))}{r}, \quad g_{tt}(r) = -e^{2A(r)}g_{rr}(r). \tag{37}$$

In the next section, we study thermodynamic behavior of 4D GB Bardeen black hole horizon with this mass distribution function.

4 Thermodynamics of 4D GB dS/AdS Bardeen black hole

For the mass distribution solution (34), the exterior horizon equation of the 4D GB dS/AdS Bardeen black hole is obtained by solving the equation $g^{rr}(r_+) = 0$ given by (37) as follows.

$$\begin{aligned} 1 =& x^2\left(\frac{2m}{(x^2+1)^{\frac{3}{2}}} - \frac{\lambda}{3}\right) \\ &+ \mu\left\{\frac{3m^2(x^4+4x^2-1)}{(x^2+1)^3} + \frac{5x^2\lambda^2}{9} + \frac{8m\lambda}{(x^2+1)^{\frac{3}{2}}} + 3m^2\frac{\tan^{-1}x}{x} - 8m\lambda\frac{\sinh^{-1}x}{x}\right\}, \end{aligned} \tag{38}$$

in which we defined the dimensionless ADM mass m and the dimensionless cosmological parameter $\hat{\lambda}$, dimensionless horizon position x and dimensionless GB parameter μ as follows.

$$m = \frac{M_{ADM}}{Q}, \quad \hat{\lambda} = \Lambda Q^2, \quad x = \frac{r_+}{Q}, \quad \mu = \frac{\tilde{\alpha}}{Q^2}. \tag{39}$$

The Hawking temperature of this kind of the black hole is obtained vs the surface gravity such that

$$T = \frac{-g'_{tt}}{4\pi}\bigg|_{r_+} = \frac{e^{2A(r_+)}}{4\pi}\left(\frac{2M(r_+)}{r_+^2} - \frac{2M'(r_+)}{r_+}\right), \tag{40}$$

which by substituting $r_+ = 2M(r_+)$, the equation $M'(r)$ given by (22) and the solution (36) reads the following dimensionless form

$$t(x) = 4\pi QT \tag{41}$$
$$= \frac{x^{1+\hat{\lambda}\mu}}{(x^2 - 6\mu)}\left[1 - \frac{3\mu}{x^2} - \frac{6mx^2}{(x^2+1)^{\frac{5}{2}}} + \hat{\lambda}x^2\right] \times \exp\left\{6\mu m\left[\tanh^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) - \frac{1}{\sqrt{x^2+1}}\right]\right\},$$

where we have substituted the dimensionless quantities (39). To study thermodynamic behavior of this kind of black hole, it is simpler to use asymptotic behavior of the event horizon equation and the Hawking temperature equation instead of their exact forms. In fact, calculations with the exact solutions take on complicated forms. To do so, we can study two different cases as $x < 1$ and $x > 1$. In the case $x > 1$ asymptotic series forms of the horizon equation and the Hawking temperature take on divergent forms and so give not suitable physical situations for the P-V curves at constant temperature and other thermodynamic diagrams. But, for small scale black holes $x < 1$ the behavior of P-V diagram predicts small to large black hole phase transition and vice versa dependent on choosing the numeric values which we use for the GB parameter μ.

According to the GB parameter μ which takes on some small values $\mu < 1$, let's to start by substituting the mass parameter $m(x)$ given by the horizon equation (38) into the temperature equation (41) which takes on a lengthy form and so we calculate its series expansion about small values of the parameters $\mu < 1$ and $x < 1$ such that

$$t \approx \frac{(213\mu - 20)}{10x} - \frac{6\mu}{x^3}\left(4 + \ln\left|\frac{2}{x}\right|\right) + \frac{2\mu(4 - \ln 2)\lambda}{x} + O(x, x\ln x, \mu^2), \tag{42}$$

and

$$m \approx \frac{5\mu}{2x^4} + \frac{A}{x^2} + B + O(x^2, \mu^2) \tag{43}$$

where we defined

$$A = \frac{1}{2} + \frac{99}{20}\mu - \lambda\mu, \quad B = \frac{3}{4} + \frac{\lambda}{6} + \frac{1629}{560}\mu - \frac{\lambda^2\mu}{3} - \frac{11\lambda\mu}{10}. \tag{44}$$

In the above series, expansion forms the $O(\cdots)$ terms are not divergent for small values of $x < 1$ and $\mu < 1$ and so we can remove them when we study thermodynamic behavior of small scale black holes. To obtain equation of state, we set thermodynamic specific volume v and dS/AdS pressures respectively as follows

$$AdS: \quad \lambda = -8\pi p, \quad v = \frac{-16\pi\mu(4 - \ln 2)}{x} \tag{45}$$

and

$$dS: \quad \lambda = 8\pi p, \quad v = \frac{16\pi\mu(4 - \ln 2)}{x}, \tag{46}$$

for which the temperature equation (42) reads

$$dS/AdS: \quad t = pv + 6\mu\epsilon\left(\frac{v}{16\pi\mu(4-\ln 2)}\right)^3\left(4+\ln\left|\frac{v}{8\pi\mu(4-\ln 2)}\right|\right) - \epsilon\frac{(213\mu-20)}{10}\left[\frac{v}{16\pi\mu(4-\ln 2)}\right] \tag{47}$$

where $\epsilon = +1(-1)$ for AdS(dS) sector of the background space time. By looking at the positivity condition of the specific volumes (45) and (46), one can infer that for AdS sector in which $x < 0$, we should choose negative magnetic charge, but for dS sector in which $x > 0$, we should choose $Q > 0$. In order to plot P-V diagrams at constant temperature, the critical points are needed to be obtained from the critical equations of $\frac{\partial t}{\partial v}|_p = 0$ and $\frac{\partial^2 t}{\partial v^2}|_p = 0$ which by substituting the equation of state (47) read

$$v_c = 8\pi\mu(4-\ln 2)e^{-\frac{29}{6}}, \quad p_c = \frac{\epsilon[45e^{-\frac{29}{3}} + 426\mu - 40]}{320\pi\mu(4-\ln 2)}, \quad t_c = \frac{\epsilon\mu e^{-\frac{29}{2}}}{2}. \tag{48}$$

It is useful to rewrite the equation of state (47) versus the following re-scaled thermodynamic variables.

$$\bar{v} = \frac{v}{v_c}, \quad \bar{p} = \frac{p}{p_c}, \quad \bar{t} = \frac{t}{t_c} \tag{49}$$

such that

$$dS/AdS: \quad \bar{t} = a(\mu)\bar{p}\bar{v} + \frac{3}{2}\bar{v}^3\left[\ln\bar{v} - \frac{5}{6}\right] + b(\mu)\bar{v} \tag{50}$$

where we defined

$$a(\mu) = \frac{p_c v_c}{t_c} = \frac{45\mu + (426\mu - 40)e^{\frac{29}{3}}}{20\mu}, \quad b(\mu) = \frac{(20-213\mu)e^{\frac{29}{3}}}{10\mu}. \tag{51}$$

This equation of state has same form for both dS and AdS sector of the spacetime. Thus, by substituting some different values of the GB parameter, we apply to plot P-V diagram at constant temperature for critical temperature $\bar{t}_c = 1$ and its below and upper values in Figure 2. To do so, we define a critical value for the GB parameter as $\mu_c = 0.09323992798$ which is calculated by substituting $\bar{v} = \bar{v}_c = 1$, $\bar{p} = \bar{p}_c = 1$ and $\bar{t} = \bar{t}_c = 1$ into the equation of state (50) and we plot several P-V diagrams at constant temperature for $\mu < \mu_c$, $\mu = \mu_c$ and $\mu > \mu_c$. These diagrams show that a large to small black hole phase transition for $\mu \geq \mu_c$ (see Figures 1-a,b,c,d) and small to large phase transition for $\mu < \mu_c$ (see Figures 2-a,b). In the Figures 1-a,b,c,d, the diagrams have a local maximum point at higher specific volume and a local minimum point at smaller specific volume. It is known that thermodynamic stability is happened in the minimum point of the P-V diagram while its instability is happened at maximum point of the diagram. By looking at the diagrams given at the Figures 1 and 2, one can infer that position of the minimum and the maximum points of the diagrams are exchanged for $\mu \geq \mu_c$ and $\mu < \mu_c$ respectively. In the next section, we investigate Joule-Thomson expansion of this kind of black hole by plotting isenthalpic $t - p$ diagrams and inversion curves.

4.1 JT Expansion

To study the Joule-Thomson adiabatic expansion of the 4D GB dS/AdS Bardeen black hole, we should investigate isenthalpic curves for all scales of the black holes $x < 1$ and $x > 1$.

Then, we can not use series solutions of the horizon equation and the Hawking temperature given in the previous section for small values $x < 1$. Therefore, we must be use exact forms of these equations but we can still use series expansions of them for small GB parameter $\mu < 1$. Hence, we first solve the horizon equation vs the cosmological parameter $\hat{\lambda}$ leading to two different solutions which one of them reaches to some physical situations where the JT expansion happens. By substituting it into the temperature equation, we can obtain a suitable form for the temperature equation to be independent of the $\hat{\lambda}$ term but having a lengthy form. Hence, we calculate its Taylor series expansion about small values of μ which up to the second order term reads

$$\hat{\lambda} \simeq \frac{6m}{(x^2+1)^{\frac{3}{2}}} - \frac{3}{x^2} - \frac{3\mu}{x^5(x^2+1)^3} \times \tag{52}$$

$$\Big[4mx(x^2+1)^{3/2}(12mx\sinh^{-1}x + 5x^2 + 6)$$

$$-3m(x^2+1)^3(8\sinh^{-1}x + mx^2\tan^{-1}x) - 5x(x^2+1)^3 - 3m^2(x^7 + \frac{32}{3}x^5 + 15x^3)\Big]$$

in which $p = -\frac{\epsilon\hat{\lambda}}{8\pi}$ with ($\epsilon = +(-)$) for AdS(dS) sector of the space time, is pressure of dS/AdS background space. Moreover, we approximate the relation of temperature for small μ as follow

$$t \simeq \left(\frac{1}{x} + \lambda x - \frac{6mx}{(x^2+1)^{\frac{5}{2}}}\right) - \mu\Bigg\{\frac{3}{x^3} - \tag{53}$$

$$\left[6m\left[\tanh^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) - \frac{1}{\sqrt{x^2+1}}\right] + \frac{6}{x^2} + \lambda \ln x\right]\left(\frac{1}{x} + \lambda x - \frac{6mx}{(x^2+1)^{\frac{5}{2}}}\right)\Bigg\}.$$

We plot the isoenthalpic T-P curves for all scales of the 4D GB dS/AdS Bardeen black holes with three chosen numeric values for dimensionless GB parameter as $\mu = 0.0001$, 0.001, and 0.01. To determine inversion curves where the cooling phase and heating phase of the black hole is separated with intersection point of the inversion curve and isenthalpic curves (see Figures 3,4,5) we should calculate the JT coefficient through $\mu_{JT} = \frac{\partial t}{\partial p}|_m$ (see appendix) and solve $\mu_{JT} = 0$ for which we obtain a lengthy relation for the inversion enthalpy (the mass m_i) and so we do not mention it in this paper. To plot inversion curves we substitute the inversion mass m_i into the relations (52) and (53) to obtain suitable relations for the inverse temperature and inversion pressure respectively vs x. Then, all isenthalpic curves together with inversion curves are plotted for different values of the horizon parameters x. These diagrams are collected in the Figures 3-a,b for $\mu = 0.0001$, the Figures 4-a,b for $\mu = 0.001$ and the Figures 5-a,b for $\mu = 0.01$ with different black hole masses $1.5 \leq m \geq 22$, respectively. One can see that left side of the intersection point of the inversion curve with the isenthalpic curves, where the slope of the curves or the JT coefficient takes on some positive values the black hole system participates in the heating phase but for the right side of the intersection point, it participates in the cooling phase. Also, one can look at the Figures 3-a,b and 4-a,b and compare them to infer that by rasing the GB parameter, the black hole system at constant enthalpy takes on other subsystem which does not participate in the JT adiabatic expansion (please compare Figures 3-a,b and Figures 4-a,b and Figure 5-a with Figure 5-b). These are shown with straight-lines. Particularly, for large values of the GB parameter (see Figure 5-b) there is not a intersection between the inversion curve (solid blue line) with the isenthalpic curves. In fact, to appear a JT adiabatic expansion in a thermodynamic system the inversion curves should intersect maximum point of the isenthalpic curves which are appeared in all Figures 3-a,b and 4-a,b and 5-a but not in the

Figure 5-b. Furthermore, we should point out that the inversion curves have two different branches which one of them intersects with the isenthalpic curves for small values mass of the black holes (see Figures 3-a, 4-a and 5-a) while for massive black hole these two branches are approaching to each other and in fact two branches of the inversion curves intersect maximum point of the isenthalpic T-P curves (see Figures 3-b and 3-b).

5 Conclusion

In this work, we have used a consistent model of Einstein-Gauss-Bonnet gravity to study thermodynamics of 4D dS/AdS GB Bardeen black hole. Metric source of this type of black hole is nonlinear electromagnetic fields with a non-vanishing magnetic charge. Physical importance of this type of black holes is nonsingular property they have and are applicable to study black hole structure of center of galaxies. We obtained analytical solution of the metric field equation for small values of the Gauss-Bonnet coupling constant. We calculated exterior horizon of this kind of black holes and the Hawking temperature. At last, by calculating the equation of state in presence of both dS and AdS space pressures, we investigated the black hole thermodynamics by focusing on the phase transitions and Joule-Thomson adiabatic expansion of the black hole in presence of Gauss Bonnet counterterms. In this way, we saw that the Gauss-Bonnet coupling constant plays important role in these phase transitions. Positivity condition on the thermodynamic volume causes to be valid negative (positive) magnetic charge for AdS (dS) space pressure. Our mathematical calculations predict that 4D AdS GB Bardeen black hole takes on small to large phase transition from small values of the Gauss-Bonnet coupling constant and vice versa for larger values of this coupling constant. Also, we understand that for small values of the Gauss-bonnet term effect the black hole has a subsystem which participates in the JT adiabatic expansion but for larger values of the Gauss-Bonnet effect there is a second subsystem for the black hole which does not participate in the cooling-heating (JT expansion) phase transition. As an extension of this work one can investigate JT expansion for other types of nonsingular magnetic charged black holes (see for instance [63,64]). As a future work, we like to investigate other thermodynamic properties of the 4D GB dS/AdS Bardeen black hole such that holographic entanglement entropy and complexity growth rate via complexity=action conjecture.

Appendix

As we mentioned previously, the JT expansion occurs at constant enthalpy $H = U + PV$ for which we can write

$$dH = TdS + VdP \tag{54}$$

where we used

$$TdS = dQ = dU + PdV. \tag{55}$$

Applying $dH = 0$ the equation (54) reduces to the form $0 = TdS + VdP$ which can be rewritten as

$$\frac{dH}{dP} = 0 = T\left(\frac{\partial S}{\partial P}\right)_H + V. \tag{56}$$

If we assume that the entropy depends to the temperature T and the pressure P as $S = S(T, P)$ then we can write $dS = \left(\frac{\partial S}{\partial P}\right)_T dP + \left(\frac{\partial S}{\partial T}\right)_P dT$ for which we will have

$$\left(\frac{\partial S}{\partial P}\right)_H = \left(\frac{\partial S}{\partial P}\right)_T + \left(\frac{\partial S}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_H. \tag{57}$$

Substituting (57) into the equation (56) we obtain

$$0 = -T\left(\frac{\partial V}{\partial T}\right)_P + C_P\left(\frac{\partial T}{\partial P}\right)_H + V, \tag{58}$$

where we used $C_P = T\big(\frac{\partial S}{\partial T}\big)_P$ and the Maxwell relationship $\big(\frac{\partial S}{\partial P}\big)_T = -\big(\frac{\partial V}{\partial T}\big)_P$ which by solving $\big(\frac{\partial T}{\partial P}\big)_H$ one can obtain the JT coefficient $\hat{\lambda}_{JT} = \big(\frac{\partial T}{\partial P}\big)_H$.

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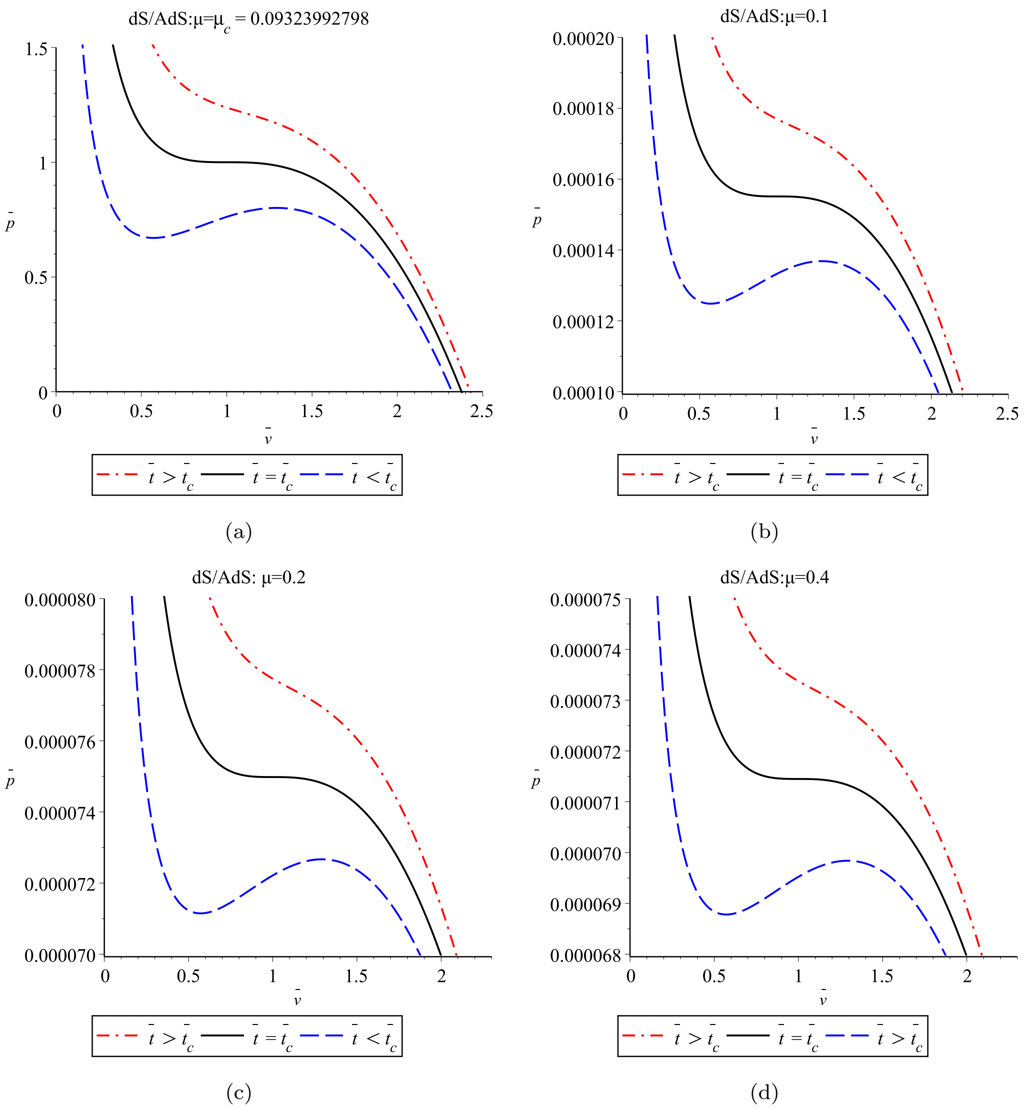
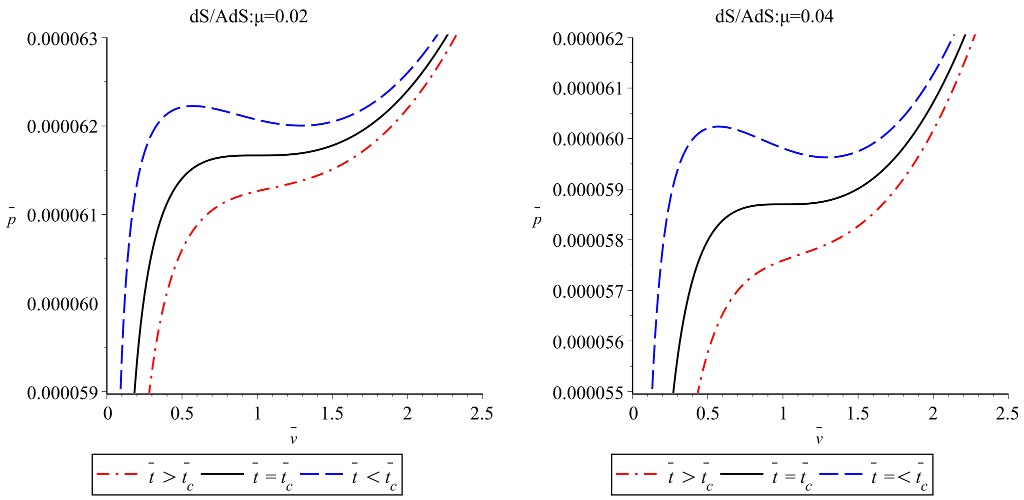
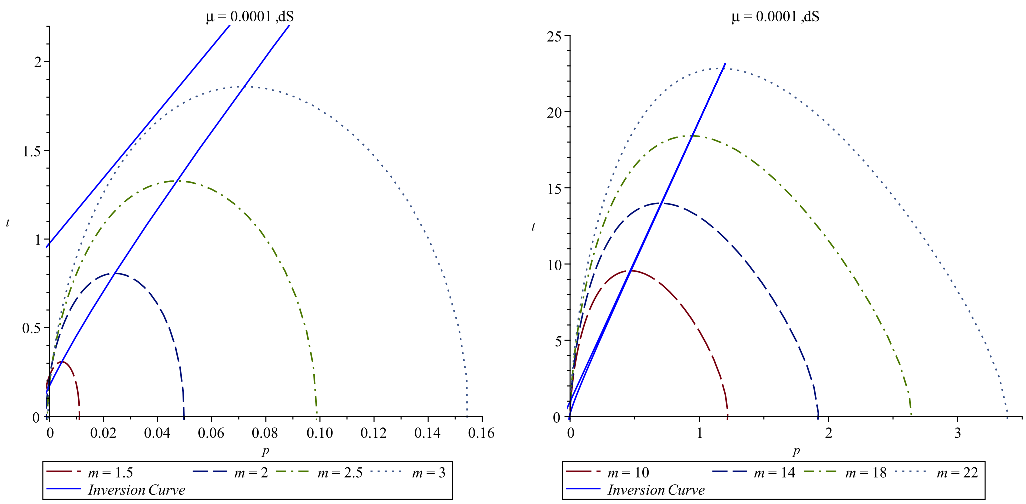


Figure 1: P-V diagrams of 4D GB dS/AdS Bardeen black holes with $\mu \geq \mu_c$



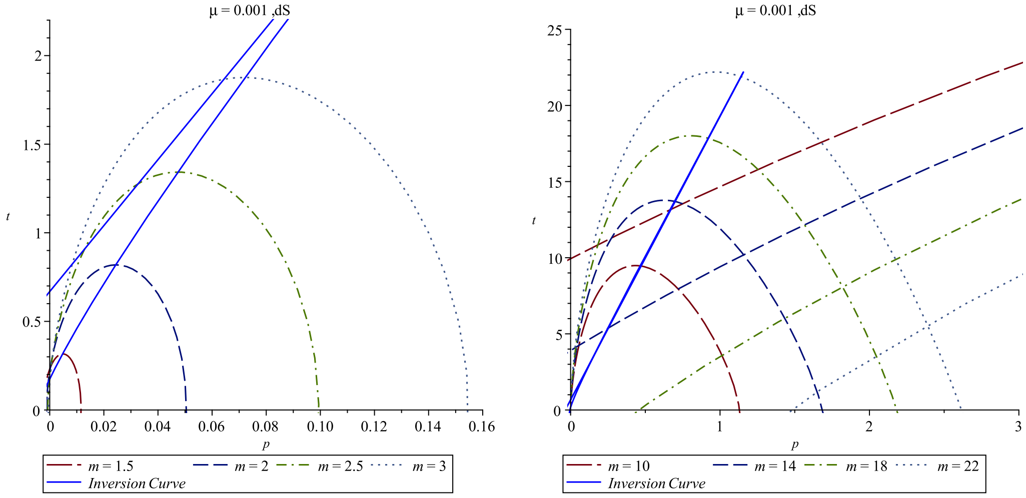
(a) (b)

Figure 2: P-V diagrams of 4D GB dS/AdS Bardeen black holes with $\mu < \mu_c$



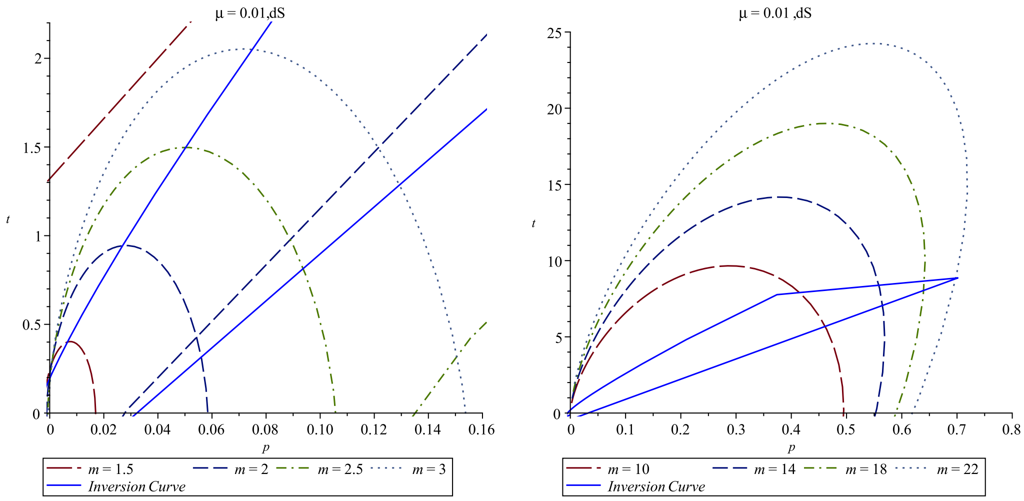
(a) (b)

Figure 3: Isenthalpic and inversion curves for 4D GB dS/AdS Bardeen black holes with $\mu = 0.0001$ and different values of the enthalpy $1.5 \leq m \leq 22$



(a) (b)

Figure 4: Isenthalpic and inversion curves for 4D GB dS/AdS Bardeen black holes with $\mu = 0.001$ and different values of the enthalpy $1.5 \leq m \leq 22$



(a) (b)

Figure 5: Isenthalpic and inversion curves for 4D GB dS/AdS Bardeen black holes with $\mu = 0.01$ and different values of the enthalpy $1.5 \leq m \leq 22$