Iranian Journal of Astronomy and Astrophysics
Vol. 12, No. 2, Spring 2025, 127–142
©Available online at http://ijaa.du.ac.ir
DOI: 10.22128/ijaa.2025.3025.1222
Online ISSN: 2383–403X

Iranian Journal of Astronomy and Astrophysics

Research Paper

Braneworld Cosmology with a non-minimally Coupled Scalar Field and Palatini Bulk Gravity

Mojtaba Alimoradi[1] · Tahereh Azizi*[2] · Narges Rashidi[3]

[1] Department of Theoretical Physics, Faculty of Science, University of Mazandaran, P. O. Box 47416–95447, Babolsar, Iran;
E-mail: moji.2002@yahoo.com

[2] Department of Theoretical Physics, Faculty of Science, University of Mazandaran, P. O. Box 47416–95447, Babolsar, Iran;
*E-mail: t.azizi@umz.ac.ir

[3] Department of Theoretical Physics, Faculty of Science, University of Mazandaran, P. O. Box 47416–95447, Babolsar, Iran;
E-mail: n.rashidi@umz.ac.ir

Received: 22 August 2025; **Accepted:** 28 Septembere 2025; **Published:** 4 October 2025

Abstract. We study a five-dimensional braneworld cosmology in the Palatini formalism, where a bulk scalar field couples non-minimally to the Ricci scalar. The bulk contains a delta-function brane with $\mathbb{Z}_2$ symmetry that confines standard matter fields. We derive generalized field equations and obtain the effective Friedmann and Raychaudhuri equations on the brane, which display key differences from the metric formulation. Non-minimal coupling of Palatini Ricci scalar and scalar field corrections significantly modify both early- and late-time dynamics: in the high-energy regime, the non-minimal coupling reshapes the effective scalar potential, altering slow-roll parameters and the duration of inflation—negative couplings flatten the slope and enhance inflation, while positive couplings steepen it, with further modulation from the bulk cosmological constant. To test the viability of the model, we constrained its parameter space using the Planck 2018 datasets to obtain the observationally viable ranges for the model's parameter. At low energies, the system approaches scalar-dominated or vacuum de Sitter solutions, where the effective Newton constant and cosmological constant emerge from the interplay of brane tension, scalar potential, and bulk contributions. This structure naturally realizes a generalized Randall–Sundrum fine-tuning and supports late-time acceleration.

Keywords: Braneworld, Cosmology, Palatini Formalism, Non-Minimal Coupling, Inflation, Dark Energy.

1 Introduction

Theories of gravitation beyond General Relativity (GR) have been extensively explored as possible frameworks to address open problems in cosmology, such as the origin of inflation, the nature of dark energy, and the unification of gravitational and quantum phenomena. Among these, the modified gravity theories in Palatini formalism have attracted a lot of

*Corresponding author

attentions. In the Palatini approach, the metric and affine connection are treated as independent variables in the action, leading to field equations that differ from those of the standard metric formalism for nonlinear extensions of GR. This framework often yields second-order field equations even for higher-order curvature invariants, thereby avoiding certain instabilities such as the Ostrogradsky ghost [1,2]. Moreover, the Palatini formalism can introduce effective matter–curvature couplings that significantly alter the dynamics of scalar fields in gravitational theories [3–5].

Scalar fields play a central role in modern cosmology, both as drivers of early-universe inflation and as candidates for late-time cosmic acceleration. When coupled minimally to curvature, scalar fields yield dynamics governed solely by their potential and kinetic terms. However, in many high-energy theories, including those arising from dimensional reduction of higher-dimensional gravity or string-inspired models, scalar fields naturally couple non-minimally to curvature via terms of the form $f(\phi)R$ [6,7], where $f(\phi)$ is an arbitrary function of the scalar field. Such couplings can arise as quantum corrections, from renormalization in curved spacetime, or from the effective low-energy limits of more fundamental theories [8,9]. Non-minimal couplings modify the gravitational sector and can generate rich phenomenology, including variations in the effective gravitational constant, new inflationary mechanisms [10,11], and late-time acceleration without introducing exotic matter [12,13].

In higher-dimensional settings, scalar fields in the bulk have been explored in a variety of contexts, from stabilization of extra dimensions [14] to brane inflation [15] and domain-wall cosmologies [16]. Minimal coupling cases have been widely examined [17,18], while non-minimally coupled bulk scalars have received comparatively less attention despite their potential for rich gravitational dynamics [19,20]. The introduction of such couplings within the Palatini framework can furthur modify the scalar field dynamics by introducing curvature-dependent terms into the effective potential.

Braneworld scenarios, inspired in part by developments in string theory and M-theory, propose that our observable universe is a $(3 + 1)$-dimensional hypersurface (brane) embedded in a higher-dimensional bulk spacetime. In these models, ordinary matter remains confined to the brane, while gravity is free to propagate through the whole spacetime. The cosmological evolution of the brane is governed by an effective Friedmann equation that accounts for the bulk's influence in a nontrivial way. The seminal Randall–Sundrum models [21,22] demonstrated that extra dimensions could be noncompact yet consistent with four-dimensional gravity at large scales. Cosmological models in this framework reveal modified Friedmann equations [23,24], high-energy corrections in the early universe, and possible explanations for late-time acceleration [25]. Extensions to include bulk scalar fields [26,27] further enrich the phenomenology, though most studies have been limited to minimal coupling and metric formulations.

More recently, a small number of works have explored modified gravity in Palatini formalism in braneworld scenarios, primarily within thick brane configurations [28–31]. While these studies provide valuable insights into the gravitational structure of such models, they focus mainly on the pure gravity sector and do not address the cosmological dynamics of thin branes. In particular, the role of a non-minimally coupled bulk scalar field in shaping the effective four-dimensional cosmology within the Palatini framework remains unexplored. The purpose of this work is to formulate and analyze such a model, deriving the bulk and brane field equations, generalized junction conditions, and examining its implications for both early- and late-time cosmic evolution.

The paper is organized as follows. In Sec. 2, we introduce the action for a scalar field non-minimally coupled to the Ricci scalar in the Palatini formalism, and derive the generalized Einstein equations, the scalar field equation of motion, and the modified junction conditions. In Sec. 3, we apply these junction conditions to obtain the effective on-brane field equations and scalar evolution. In Sec. 4, we investigate the resulting cosmological dynamics at early and late times, providing a unified framework for inflation and late-time acceleration. Finally, Sec. 5 summarizes our main results. Throughout this work, capital Latin letters $\{M, N, \cdots\}$ denote the five-dimensional coordinate indices running from 0 to 4 and Greek letters $\{\mu, \nu, \cdots\}$ denote the four-dimensional coordinates running from 0 to 3.

2 Setup and Field Equations

We consider the five-dimensional action

$$S = \int d^5x\sqrt{-g}\left[\frac{f(\phi)}{2\kappa_5^2}\mathcal{R}(\Gamma) - \frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V(\phi) - \Lambda_5\right] + \int_\Sigma d^4x\sqrt{-h}(\mathcal{L}_{\rm brane} - \lambda), \quad (1)$$

where $\sqrt{-g}$ is the determinant of the metric g_{MN}, $\kappa_5^2 = 8\pi G_5$ is the gravitational coupling, and Λ_5 denotes the bulk cosmological constant. The framework is set in $D = 5$ with signature $(-++++)$, a torsion-free independent connection $\Gamma^P{}_{MN}$ (Palatini formalism), and a thin brane Σ with induced metric $h_{\mu\nu}$. The parameter λ is the brane tension, while $\mathcal{L}_{\rm brane}$ represents the ordinary matter sector localized on the brane. The Ricci scalar $\mathcal{R}(\Gamma)$ is constructed solely from the independent connection Γ^C_{AB}. The brane is placed at $x_5 \equiv y = 0$, modeled by a delta-function distribution and subject to a $\mathbb{Z}_2$ symmetry across it. The scalar field ϕ couples to $\mathcal{R}(\Gamma)$ through a general function $f(\phi)$, allowing for rich phenomenology in both cosmology and gravity.

Varying the action (1) with respect to the metric yields the generalized Einstein equations

$$f(\phi)\,\mathcal{R}_{MN} - \tfrac{1}{2}f(\phi)\,\mathcal{R}\,g_{MN} = \kappa_5^2\,T_{MN}, \quad (2)$$

where $T_{MN} = T^{(\rm bulk)}_{MN} + T^{(\rm brane)}_{MN}$. The bulk energy-momentum tensor is defined as

$$T^{(\rm bulk)}_{MN} = \partial_M\phi\,\partial_N\phi - \tfrac{1}{2}g_{MN}\,\partial^P\phi\,\partial_P\phi - g_{MN}\big(V(\phi) + \Lambda_5\big). \quad (3)$$

The brane contribution to the energy-momentum tensor takes the form

$$T^{(\rm brane)}_{MN} = S_{\mu\nu}\,e^\mu{}_M e^\nu{}_N\,\delta(\Sigma), \quad (4)$$

where $e^\mu{}_M$ are the tangential projectors and we have defined

$$S_{\mu\nu} \equiv -\frac{2}{\sqrt{-h}}\frac{\delta}{\delta h^{\mu\nu}}\int_\Sigma d^4x\,\sqrt{-h}\,\mathcal{L}_{\rm brane}\,. \quad (5)$$

For a perfect fluid localized on the brane with energy density ρ_b and pressure p_b, we have

$$S_{\mu\nu} = -\lambda\,h_{\mu\nu} + \tau_{\mu\nu}\,, \quad (6)$$

with

$$\tau_{\mu\nu} = (\rho_b + p_b)\,u_\mu u_\nu + p_b\,h_{\mu\nu}\,. \quad (7)$$

Here, we assume that the perfect fluid components have no explicit dependence on ϕ.

Contracting Eq. (2), we obtain

$$T = -\frac{3f(\phi)}{2\kappa_5^2}\,\mathcal{R}, \quad (8)$$

where T is the trace of the energy-momentum tensor. This relation shows that the gravitational sector and the matter sector are connected through an algebraic relation. Varying the action (1) with respect to the scalar field yields the scalar field equation of motion,

$$\nabla^2\phi - V_{,\phi}(\phi) + \frac{1}{2\kappa_5^2}\, f_{,\phi}(\phi)\,\mathcal{R} = 0, \tag{9}$$

where ∇ denotes the covariant derivative with respect to the metric g_{MN}. Using Eqs. (8) and (9), we find

$$\nabla^2\phi - V_{,\phi}(\phi) - \frac{1}{3f(\phi)}\, f_{,\phi}(\phi)\, T = 0\,. \tag{10}$$

The above equation implies that, in non-minimally coupled Palatini braneworlds, the brane, bulk, and scalar field are dynamically coupled. In particular, the non-minimal Palatini interaction induces an exchange of energy between the brane sector and the bulk scalar field.

Variation with respect to the independent connection yields the compatibility condition

$$\nabla^\Gamma_P \left(\sqrt{-g} f(\phi) g^{MN}\right) = 0. \tag{11}$$

where ∇^Γ_P denotes the covariant derivative associated with Γ^P_{MN}. Introducing the conformally related metric

$$q_{MN} = f(\phi)^{\frac{2}{3}} g_{MN}, \tag{12}$$

one finds that

$$\sqrt{-q}\, q^{MN} = \sqrt{-g}\, f(\phi)\, g^{MN}. \tag{13}$$

Equation (11) then implies

$$\nabla^{(\Gamma)}_C (\sqrt{-q}\, q^{AB}) = 0\,, \tag{14}$$

which shows that Γ is the Levi-Civita connection of the conformal metric q_{MN}. Consequently, the Palatini Ricci tensor can be expressed in terms of the metric Ricci tensor $R_{MN}(g)$ as

$$\mathcal{R}_{MN} = R_{MN}(g) - \frac{1}{f(\phi)}\nabla_M\nabla_N f(\phi) + \frac{4}{3f(\phi)^2}\nabla_M f(\phi)\nabla_N f(\phi) - \frac{1}{3f(\phi)} g_{MN}\Box f(\phi) \tag{15}$$

and the Palatini Ricci scalar becomes

$$\mathcal{R} = R(g) - \frac{8}{3f(\phi)}\Box f(\phi) + \frac{4}{3f(\phi)^2}\nabla_M f(\phi)\nabla^M f(\phi), \tag{16}$$

where ∇_M is the covariant derivative compatible with g_{MN}, $\Box \equiv g^{MN}\nabla_M\nabla_N$ is the corresponding d'Alembertian, and $R(g)$ denotes the Ricci scalar constructed from the spacetime metric g_{MN}. Using the above relations, the Palatini field equations (2) can be recast into the form of the usual Einstein equations with an effective source,

$$G_{MN} = R_{MN} - \frac{1}{2}R\, g_{MN} = \frac{\kappa_5^2}{f(\phi)} T^{(\text{eff})}_{MN}, \tag{17}$$

where T^{eff}_{MN} contains the modifications induced by the non-minimal coupling, together with the bulk and brane matter contributions defined in (3) and (5), and is given by

$$\begin{aligned} T^{(\text{eff})}_{MN} =& \frac{1}{\kappa_5^2}\left[\nabla_M\nabla_N f(\phi) - g_{MN}\Box f(\phi) - \frac{4}{3f(\phi)}\nabla_M f(\phi)\nabla_N f(\phi) + \frac{2}{3f(\phi)} g_{MN}\nabla_P f(\phi)\nabla^P f(\phi)\right] \\ &+ T_{MN} + T^{(\text{brane})}_{MN}. \end{aligned} \tag{18}$$

Because of the delta-function source (thin brane formalism) and the imposed $\mathbb{Z}_2$ symmetry, the metric and scalar field must satisfy appropriate matching conditions. The distributional part of Eq. (17) leads to the generalized Israel junction conditions,

$$\left[K_{\mu\nu} - K\,h_{\mu\nu}\right]_0 = -\frac{\kappa_5^2}{f(\phi_0)}\left(S_{\mu\nu} - \frac{1}{3}h_{\mu\nu}S\right), \tag{19}$$

where $K_{\mu\nu}$ is the extrinsic curvature of the brane, $S = S^{\mu}{}_{\mu}$ is the trace of the brane energy-momentum tensor, and

$$[A]_0 \equiv A(0^+) - A(0^-),$$

denotes the jump of a quantity A across the brane. For a $\mathbb{Z}_2$-symmetric brane, one has $[K_{\mu\nu}] = 2K_{\mu\nu}^{(+)}$. Furthermore, integrating the Palatini scalar equation of motion across the brane yields a relation between the jump of the normal derivative of ϕ and the jump of the trace of the extrinsic curvature. In compact form, this reads

$$[\phi']_0 + \frac{f_{,\phi}(\phi_0)}{\kappa_5^2}\,[K] = 0, \tag{20}$$

where $\phi_0 \equiv \phi(0,t)$ denotes the scalar field evaluated on the brane. It should be emphasized that the brane energy-momentum tensor does not contain independent scalar-field contributions; hence, the scalar jump is induced purely through geometric effects of the Palatini non-minimal coupling.

3 Equations of Motion on the Brane

Since we are interested in cosmological solutions, we consider the following metric in Gaussian normal coordinates,

$$ds^2 = -n^2(y,t)dt^2 + a^2(y,t)\gamma_{ij}dx^i dx^j + dy^2, \tag{21}$$

where γ_{ij} is the metric of a maximally symmetric three-space with curvature $k = 0, \pm1$, and the gauge choice $n(0,t) = 1$ has been imposed. The extra-dimensional coordinate is y, with the brane located at $y = 0$. From the metric (21), we obtain the nonvanishing components of the Einstein tensor as

$$G_{tt} = 3n^2\left[H^2 + \frac{k}{a^2} - \frac{a''}{a} - F^2\right], \tag{22}$$

$$G_{ty} = 3\,\frac{\dot{a}\,n' - n\,\dot{a}'}{a\,n}, \tag{23}$$

$$G_{yy} = 3\left[F^2 - \frac{k}{a^2} - H^2 + H\frac{\dot{n}}{n^2} - \frac{\ddot{a}}{a\,n^2}\right], \tag{24}$$

$$G_{ij} = g_{ij}\left[F^2 + 2F\frac{n'}{n} - H^2 + 2H\frac{\dot{n}}{n^2} - 2\frac{\ddot{a}}{a\,n^2} + \frac{k}{a^2}\right], \tag{25}$$

where a dot and a prime denote differentiation with respect to the time coordinate t and the fifth coordinate y, respectively. We have also defined $H \equiv \frac{\dot{a}}{a\,n}$ and $F \equiv \frac{a'}{a}$. The components of the energy-momentum tensor of the bulk (3) are then found to be

$$T_{tt} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}n^2\phi'^2 + n^2\,(V + \Lambda_5), \tag{26}$$

$$T_{yy} = \frac{1}{2}\phi'^2 + \frac{1}{2}\frac{\dot\phi^2}{n^2} - (V + \Lambda_5), \tag{27}$$

$$T_{ij} = g_{ij}\left(\frac{1}{2}\frac{\dot\phi^2}{n^2} - \frac{1}{2}\phi'^2 - (V + \Lambda_5)\right), \tag{28}$$

$$T_{ty} = \dot\phi\,\phi', \tag{29}$$

under the assumption that $\phi = \phi(t, y)$. The brane energy-momentum is localized on the hypersurface $y = 0$, and the nonvanishing components of the tensor (5) are

$$T^{(\text{brane})}_{tt} = (\rho_b + \lambda)\,\delta(y), \qquad T^{(\text{brane})}_{ij} = (p_b - \lambda)\,h_{ij}\,\delta(y), \tag{30}$$

where the brane energy density and pressure depend only on time, and $h_{ij} = a_0^2(t)\,\gamma_{ij}$ is the induced spatial metric on the brane. Plugging the above equations into the field equations (17), we obtain the equations of motion in the following form

$$3n^2\left(H^2 + \frac{k}{a^2} - \frac{a''}{a} - F^2\right) + 3\left(\frac{\dot a}{a}\frac{\dot f}{f} - n^2\frac{a'}{a}\frac{f'}{f}\right) - n^2\frac{f''}{f} + 2\frac{\dot f^{\,2}}{f^2} - \frac{2n^2}{3}\frac{f'^2}{f^2}$$
$$= \frac{\kappa_5^2}{f}\left[\tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}n^2\phi'^2 + n^2\big(V + \Lambda_5\big)\right] + \frac{\kappa_5^2}{f_0}\,(\rho_b + \lambda)\,\delta(y), \tag{31}$$

$$3f\left(\frac{\dot a}{a}\frac{n'}{n} - \frac{\dot a'}{a}\right) - \dot f' + \frac{\dot n}{n}f' + \frac{4}{3f}\,\dot f\,f' = \kappa_5^2\,\dot\phi\,\phi', \tag{32}$$

$$3f\left[F^2 - \frac{k}{a^2} - H^2 + H\frac{\dot n}{n^2} - \frac{\ddot a}{a\,n^2}\right] - \frac{1}{n^2}\left(\ddot f - \frac{\dot n}{n}\dot f + 3\frac{\dot a}{a}\dot f - \frac{2}{3f}\dot f^{\,2}\right) - \frac{n'f'}{n} + 3\frac{a'}{a}f' + \frac{2}{3f}f'^2$$
$$= \kappa_5^2\left(\tfrac{1}{2}\phi'^2 + \tfrac{1}{2}\frac{\dot\phi^{\,2}}{n^2} - (V + \Lambda_5)\right), \tag{33}$$

$$\gamma_{ij}\left\{f\left[F^2 + 2F\frac{n'}{n} - H^2 + 2H\frac{\dot n}{n^2} - 2\frac{\ddot a}{a\,n^2} + \frac{k}{a^2}\right] - \frac{1}{n^2}\left(\ddot f - \frac{\dot n}{n}\dot f + 2\frac{\dot a}{a}\dot f - \frac{2}{3}\frac{\dot f^2}{n^2}\right)\right.$$
$$\left. - \left(nn'f' - 2\frac{a'}{a}f' - f'' + \tfrac{2}{3}f'^2\right)\right\} = \gamma_{ij}\,\kappa_5^2\left[\tfrac{1}{2}\frac{\dot\phi^2}{n^2} - \tfrac{1}{2}\phi'^2 - (V + \Lambda_5) + \frac{f}{f_0}(p_b - \lambda)\,\delta(y)\right], \tag{34}$$

where $f_0 \equiv f(\phi(0, t))$. On the other hand, the equation of motion for the scalar field takes the form

$$-\frac{1}{n^2}\left(\ddot\phi - \frac{\dot n}{n}\dot\phi + 3\frac{\dot a}{a}\dot\phi\right) + \phi'' + \left(\frac{n'}{n} + 3\frac{a'}{a}\right)\phi' - V_{,\phi}(\phi) - \frac{f_{,\phi}(\phi)}{3f(\phi)}\left[\frac{3}{2}\frac{\dot\phi^2}{n^2} - \frac{3}{2}\phi'^2 - 5\big(V(\phi) + \Lambda_5\big)\right] = 0. \tag{35}$$

Now we consider the junction conditions on the brane. For the metric (21), the nontrivial components of the extrinsic curvature are ${K^t}_t = \frac{n'}{n}$ and ${K^i}_j = \frac{a'}{a}\,\delta^i_j$. Applying the generalized Israel conditions (19) to the temporal and spatial components gives

$$\left[\frac{a'}{a}\right]_0 = -\frac{\kappa_5^2}{6f_0}\,(\rho_b + \lambda), \tag{36}$$

$$\left[\frac{n'}{n}\right]_0 = \frac{\kappa_5^2}{6f_0}\,(2\rho_b + 3p_b - \lambda). \tag{37}$$

Finally, from the scalar junction condition (20), we obtain

$$[\phi']_0 = \frac{f_{,\phi}(\phi_0)}{3f_0}\,(\rho_b - 3p_b + 4\lambda), \tag{38}$$

where $f_0 \equiv f(\phi_0)$.

Evaluating the (tt) component of the field equations on the brane and applying the junction conditions, yields the following Friedmann equations:

$$H_0^2 = \frac{\kappa_5^4}{36\,f_0^2}\,(\rho_b+\lambda)^2 + \frac{\kappa_5^2}{6f_0}\left[\frac{1}{2}\dot{\phi}_0^2 + V_0 + \Lambda_5\right] + \frac{C}{a_0^4} - \frac{k}{a_0^2}, \tag{39}$$

and

$$\frac{\ddot{a}}{a} = -\frac{\kappa_5^4}{36f_0^2}(\rho_b+\lambda)(\rho_b+3p_b-2\lambda) - \frac{\kappa_5^2}{3f_0}\Big(\dot{\phi}_0^2 - V_0) - \Lambda_5\Big) - \frac{C}{a_0^4}\,. \tag{40}$$

Here, $a_0(t) \equiv a(y=0,t)$, $V_0 = V(\phi_0)$, $H_0 = \dot{a}_0/a_0$, and C/a_0^4 represents the "dark radiation" term arising from the bulk Weyl tensor. Note that the *non-minimal coupling* f_0 modulates the effective gravitational strength on the brane, effectively replacing G_5 by G_5/f_0 and altering the relative contributions of matter and scalar fields. If the bulk scalar is frozen ($\dot{\phi} = \phi' = 0$), such that $f(\phi) \equiv f_0$ remains constant on and near the brane, the scalar energy-momentum tensor reduces to

$$T_{MN}^{(\phi)} = -\big(V_0 + \Lambda_5\big)\,g_{MN}.$$

For a flat vacuum brane ($\rho_b = 0$, $k = 0$, $\mathcal{C} = 0$, $H_0 = 0$), equation (39) yields

$$V_0 + \Lambda_5 = -\frac{\kappa_5^2}{6\,f_0}\,\lambda^2. \tag{41}$$

This relation corresponds to the Randall–Sundrum fine-tuning condition, modified by the replacement $\kappa_5^2 \to \kappa_5^2/f_0$.

In addition to the Friedmann equations, the brane continuity equation is modified by the Palatini coupling and takes the form

$$\dot{\rho}_b + 3H_0(\rho_b + p_b) = Q(t). \tag{42}$$

where $Q(t) = -\frac{f_{,\phi}(\phi_0)}{f_0}\,\big(\rho_b - 3p_b + 4\lambda\big)\,\dot{\phi}_0$. This equation has important implications: even if ρ_b is initially large and matter-dominated, energy can flow between the brane fluid and the bulk scalar. This exchange modifies the redshifting of ρ_b, thereby shifting the epoch when the quadratic ρ_b^2 term becomes subdominant and the late-time scalar/tension regime is reached. Meanwhile, for radiation ($p_b = \rho_b/3$), the matter-only source term vanishes, and the radiation redshifts as usual (unless the tension term is included). For dust ($p_b = 0$), the source term acts as a multiplicative correction to the standard dilution law and can either slow down or speed up the dilution depending on the sign of $\frac{f_{,\phi}(\phi_0)}{f_0}$.

4 Cosmological evolution

To analyze the cosmological evolution, we assume that the bulk fields vary slowly in the normal direction near the brane (i.e., bulk gradients are small compared to the on-brane curvature scales). Under this assumption, the equation of motion for the scalar field can be recast in the following form

$$\ddot{\phi}_0 + 3H_0\dot{\phi}_0 + V_{,\phi}(\phi_0) = \frac{f_{,\phi}(\phi_0)}{3f_0}\Big(\tfrac{3}{2}\dot{\phi}_0^2 - 5\big(V_0 + \Lambda_5\big)\Big). \tag{43}$$

We now turn to the study of the cosmological dynamics on the brane, focusing separately on the early- and late-time regimes.

4.1 Early-time cosmology

In braneworld cosmology, the early-time regime typically corresponds to $\rho_b \gg \lambda$ and, in some cases, $\frac{\kappa_5^4\rho_b^2}{36f_0^2} \gg \frac{\kappa_5^2 V_0}{6f_0}$, so that the quadratic term dominates the Friedmann equation. Regarding the possibility of inflation at early times, there are two distinct mechanisms by which accelerated expansion may occur:

I. *High-energy brane-driven inflation:* If ρ_b is dominated by a brane scalar or fluid with an equation of state yielding negative effective pressure in the Raychaudhuri equation, the ρ_b^2 term can more readily drive inflationary expansion than in standard four-dimensional cosmology. However, the time dependence of f_0 alters the required energy scale.

II. *Bulk scalar-driven inflation:* If the bulk scalar field (evaluated on the brane) possesses a large potential energy, e.g. $V(\phi_0) \sim m\phi_0^2$, and satisfies slow-roll conditions ($\dot{\phi}_0^2 \ll V$), then the scalar contribution in the Friedmann equation can dominate and lead to inflation. In this case, the non-minimal coupling Palatini source terms on the right-hand side of (43) may modify the slow-roll conditions.

4.1.1 Dominant brane-matter:

If $\rho_b \gg \lambda$ and dark radiation is negligible, the Friedmann equation reduces to

$$H_0^2 \simeq \frac{\kappa_5^4}{36f_0^2}\,\rho_b^2\,. \tag{44}$$

Since f_0 depends on ϕ_0, the coefficient is time-dependent if ϕ_0 evolves. Using the continuity equation (42) for a perfect fluid with equation-of-state parameter w, i.e. $p_b = w\rho_b$, we obtain

$$\dot{\rho}_b + 3H_0(1+w)\rho_b = \frac{f_{,\phi}}{f_0}\,\dot{\phi}_0\,[(1-3w)\rho_b + 4\lambda]\,. \tag{45}$$

For radiation with $w = 1/3$, the matter-only source term on the right-hand side of (45) vanishes, and one finds $\rho_b \propto a^{-4}$, which is identical to the usual scaling in standard four-dimensional cosmology. However, when the brane tension is included in the exchange term, the continuity equation acquires an additional source, $Q(t) = \frac{f_{,\phi}}{f_0}\,4\lambda\,\dot{\phi}_0$, so that even radiation can exchange energy with the bulk scalar. In this case, the redshifting of radiation deviates from the pure a^{-4} law, unlike the standard 4D case, where radiation dilution is unaffected by additional couplings. Thus, the presence of the Palatini coupling allows the brane tension to mediate energy transfer between the bulk scalar and radiation, modifying its cosmological evolution.

In the case of dust matter with $w = 0$ and $\rho_b \gg \lambda$, the continuity equation can be rewritten as

$$\frac{d\ln\rho_b}{dt} = -3H_0 + \frac{f_{,\phi}}{f_0}\,\dot{\phi}_0. \tag{46}$$

If ϕ_0 evolves slowly, one can integrate the above equation approximately to obtain

$$\rho_b(t) \approx \rho_{b,*}\, a^{-3}(t)\, \exp\left(\int_{t_*}^{t} \frac{f_{,\phi}}{f_0}\, \dot{\phi}_0\, dt \right) = \rho_{b,*}\, a^{-3}(t)\, \frac{f_0(t)}{f_0(t_*)}, \tag{47}$$

i.e. $\rho_b \propto a^{-3} f_0$ if $f_{,\phi}\dot{\phi}_0$ varies slowly.

In standard four-dimensional cosmology without Palatini coupling, dust dilutes purely as $\rho_b \propto a^{-3}$. Here, however, the additional factor of $f_0(t)$ modifies this scaling: if f_0 increases with time, dilution is slowed down, while if f_0 decreases, dilution is accelerated. Thus, the scalar field ϕ effectively controls the rate of matter dilution on the brane.

4.1.2 Scalar-dominated early Universe

If the bulk scalar (evaluated on the brane) possesses a large potential energy and evolves under slow-roll conditions, the scalar contribution in the Friedmann equation can dominate and drive inflation. However, the Palatini source terms appearing in the evolution equations modify the usual slow-roll dynamics. In particular, if the scalar potential energy dominates, equation (39) reduces to

$$H_0^2 \simeq \frac{\kappa_5^2}{6 f_0}\, V_0, \tag{48}$$

so that the slow-roll conditions are altered by the Palatini source term in (43).

Assuming the standard slow-roll approximations ($\dot{\phi}_0^2 \ll V$, $|\ddot{\phi}_0| \ll H_0|\dot{\phi}_0|$), the scalar field equation can be written in the approximate form

$$3H_0\dot{\phi} + V_{,\phi} \simeq -\frac{5}{3}\frac{f_{,\phi}}{f}\left(V_0 + \Lambda_5\right), \tag{49}$$

Hence, the effective slope driving ϕ acquires an additional term $-\frac{5}{3}\frac{f_{,\phi}}{f}(V_0 + \Lambda_5)$. This modification affects both the slow-roll conditions and the number of e-folds. We define the slope of the potential as follows:

$$S(\phi) \;\equiv\; V_{,\phi} + \frac{5}{3}\frac{f_{,\phi}}{f}\left(V_0 + \Lambda_5\right). \tag{50}$$

Under this definition, the time derivative of the field is approximately $\dot{\phi} \simeq -S/(3H_0)$. We now consider the Hubble slow-roll parameters, defined as

$$\epsilon \equiv -\frac{\dot{H}_0}{H_0^2}, \qquad \eta \equiv \frac{1}{H_0}\frac{\ddot{\phi}}{\dot{\phi}}\,. \tag{51}$$

From the scalar-dominated limit $\dot{H}_0 \simeq -(\kappa_5^2/6f)\dot{\phi}^2$, and Eq. (48), it follows that

$$\epsilon \;\simeq\; \frac{\dot{\phi}^2}{V_0} \;\simeq\; \frac{2f}{3\kappa_5^2}\,\frac{S(\phi)^2}{V(\phi)^2}\,. \tag{52}$$

To estimate η, we differentiate the slow-roll relation which leads to

$$\eta \;\simeq\; \epsilon + \frac{2f_0}{\kappa_5^2}\,\frac{S_{,\phi}}{V_0}. \tag{53}$$

As a specific example, we consider a quadratic form for the non-minimal coupling function, $f(\phi) = 1 + \alpha\phi^2$, and assume the scalar potential to be of the form $V(\phi) = m\phi^2$. The slope of the potential is therefore given by

$$S(\phi) = 2m\phi + \frac{5}{3}\frac{2\alpha\phi}{1+\alpha\phi^2}\big(m\phi^2 + \Lambda_5\big). \tag{54}$$

Substituting these expressions into (52) and (53) yields closed forms for the slow-roll parameters (ϵ, η). Inflation ends when $\epsilon(\phi_{\rm end}) \simeq 1$. The number of e–folds is defined by

$$N \;\equiv\; \int_{t_{\rm ini}}^{t_{\rm end}} H_0\, dt \simeq \int_{\phi_{\rm end}}^{\phi_{\rm ini}} \frac{H_0}{|\dot{\phi}|}\, d\phi \simeq \int_{\phi_{\rm end}}^{\phi_{\rm ini}} \frac{\kappa_5^2\, V_0}{2\, f_0\, S(\phi)}\, d\phi. \tag{55}$$

For $V_0 = m\phi_0^2$ and $f_0 = 1 + \alpha\phi_0^2$, the integrand is a rational function of $\phi_0(V_0 + \Lambda_5)$ through $S(\phi)$, allowing analytic estimates in relevant parameter regimes (e.g., $|\alpha|\phi_0^2 \ll 1$ or $|\alpha|\phi_0^2 \gg 1$).

Small coupling regime $|\alpha|\phi_0^2 \ll 1$. Keeping only the leading terms, we find

$$S(\phi_0) \simeq 2m\phi_0 + \frac{10}{3}\alpha\phi_0\,(m\phi_0^2 + \Lambda_5), \tag{56}$$

$$\epsilon \simeq \frac{2}{3\kappa_5^2}\frac{\big[2m\phi_0 + \frac{10}{3}\alpha\phi_0(m\phi_0^2 + \Lambda_5)\big]^2}{m^2\phi_0^4}, \tag{57}$$

$$\eta \simeq \epsilon + \frac{2}{\kappa_5^2 m\phi_0^2}\Big[2m + \tfrac{10}{3}\alpha(3m\phi_0^2 + \Lambda_5)\Big]. \tag{58}$$

The sign of the non-minimal coupling constant α directly controls the evolution. A *negative* α reduces the effective slope $S(\phi_0)$ and thus ϵ, prolonging inflation, whereas a positive α steepens the roll.

Strong–coupling tail $|\alpha|\phi_0^2 \gg 1$. In this regime, we have $f \simeq \alpha\phi_0^2$ and $f_{,\phi}/f \simeq 2/\phi_0$, which leads to

$$H_0^2 \simeq \frac{\kappa_5^2}{6}\,\frac{m}{\alpha}, \tag{59}$$

that is quasi–constant. Additionally, we obtain the following results

$$S(\phi_0) = \frac{16}{3}m\phi_0 + \frac{10}{3}\frac{\Lambda_5}{\phi_0}, \tag{60}$$

$$\epsilon = \frac{2\alpha}{3\kappa_5^2}\,\frac{\big(16m + 10\Lambda_5/\phi_0^2\big)^2}{9m^2}\,. \tag{61}$$

If $\alpha < 0$, ϵ is suppressed and a quasi–de Sitter phase arises (H_0^2 nearly constant). For $\alpha > 0$, ϵ is enhanced, causing inflation to end more rapidly and thus narrowing the viable slow-roll window. To assess the inflationary viability, we note that the condition $\epsilon \ll 1$ together with (55) determines the admissible initial field range for given parameters $(\alpha, m, \Lambda_5, \kappa_5)$. Qualitatively, a negative α *flattens* the effective slope via $S(\phi)$ and increases N for a fixed $\phi_{\rm ini}$, whereas a positive α *steepens* the roll and reduces N. Furthermore, a negative Λ_5 lowers $S(\phi_0)$ through the $(5/3)(f_{,\phi}/f_0)\Lambda_5$ term, which aids the slow-roll conditions, while a positive Λ_5 has the opposite effect. In all cases, inflation ends when $\epsilon(\phi_{\rm end}) \simeq 1$.

Moreover, the appropriate behavior of the slow-roll parameters leads to observationally consistent values for the perturbation parameters. This provides a useful criterion for testing the viability of the model. In this regard, we consider the following relations for the scalar spectral index and the tensor-to-scalar ratio

$$n_s = 1 - 6\epsilon + 2\eta\,, \tag{62}$$

$$r = 16\epsilon\,, \tag{63}$$

where the parameters ϵ and η in our model are given by equations (57) and (58). Note that, in general, the tensor-to-scalar ratio is expressed as $r = 16\epsilon c_s$, with c_s being the sound speed. However, under the slow-roll approximation, we may set $c_s = 1$, which reduces the expression to $r = 16\epsilon$. The parameters r and n_s are crucial since they are tightly constrained by observational data. The constraint on n_s reported by the Planck 2018 TT, TE, EE+lowE+lensing+BAO+BK14 dataset is $n_s = 0.9658 \pm 0.0038$, within the ΛCDM+r+$\frac{dn_s}{d\ln k}$ framework [32]. This dataset also provides an upper bound on the tensor-to-scalar ratio as $r < 0.072$ [32]. A stronger upper limit, $r < 0.036$, has been obtained from the Planck 2018 TT, TE, EE+lowE+lensing+BAO+BK18 dataset [33]. These constraints allow us to test our model against observations and derive bounds on its parameter space.

To perform the observational analysis, we first use equation (55) to express the scalar field in terms of the e-folding number N. Substituting this into equations (57) and (58), we obtain ϵ and η as functions of N. Finally, we use these relations to evaluate (62) and (63) numerically. Based on the observational data, we identify the viable domain of the parameters N and α, illustrated in Figure 1. As shown, the model is consistent with observational constraints in certain regions of its parameter space. This represents an improvement compared to the simple single-field inflation model with a quadratic potential. While the quadratic potential in the standard single-field scenario is observationally disfavored [32], our non-minimal Palatini framework with the same potential remains consistent with data. We also derive constraints on the parameter α for several sample values of N, which are summarized in Table 1.

Table 1: Constraints on the parameter α for some sample values of N, which lead to the observationally viable values of the scalar spectral index and tensor-to-scalar ratio. These constraints are based on Planck2018 TT, TE, EE+lowE+lensing+BAO+BK14 data and Planck2018 TT, TE, EE+lowE+lensing+BAO+BK18 data.

N	Allowed range of α
40	$-8.3\times10^{-3} \le \alpha \le -6.0\times10^{-3}$
50	$-6.8\times10^{-3} \le \alpha \le -4.2\times10^{-3}$
60	$-5.6\times10^{-3} \le \alpha \le -3.4\times10^{-3}$
70	$-4.8\times10^{-3} \le \alpha \le -2.9\times10^{-3}$

4.2 Late-time cosmology

In the late-time (low-energy) regime, where $\rho_b \ll \lambda$ and dark radiation is negligible, the system often converges to a scalar-dominated or effective vacuum de Sitter solution.

4.2.1 Low-Energy Limit and Effective 4D Cosmology.

In the low-energy regime, where the matter energy density on the brane is much smaller than the brane tension, we can expand the quadratic term in the Friedmann equation (39) as

$$\frac{\kappa_5^4}{36\,f_0^2}(\rho+\lambda)^2 = \frac{\kappa_5^4\lambda^2}{36\,f_0^2} + \frac{\kappa_5^4\lambda}{18\,f_0^2}\,\rho + \mathcal{O}(\rho^2)\,. \tag{64}$$

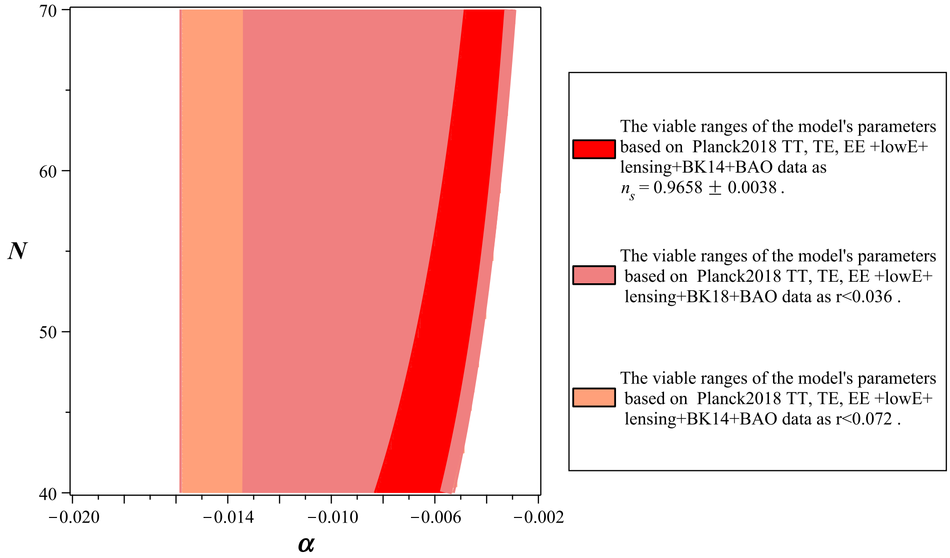

Figure 1: The observationally viable region of N and α, leading to the observationally viable values of the scalar spectral index and tensor-to-scalar ratio, based on different datasets.

The first term of the above equation acts as an effective 4D cosmological constant, while the second term reproduces the usual linear-in-ρ behavior of standard Friedmann cosmology. One can thus identify:

$$\Lambda_4^{(\mathrm{eff})} = \frac{\kappa_5^4 \lambda^2}{12\, f(\phi_0)^2} + \frac{\kappa_5^2}{2 f_0}\left[V_0 + \Lambda_5\right]. \tag{65}$$

Here, $\Lambda_4^{(\mathrm{eff})}$ plays the role of the effective 4D cosmological constant observed on the brane, incorporating contributions from both the brane tension and the bulk scalar/bulk cosmological constant. Furthermore, the effective Newton's constant on the brane, $G_4^{(\mathrm{eff})}$, can be defined as

$$8\pi G_4^{(\mathrm{eff})} = \frac{\kappa_5^4 \lambda}{6\, f_0^2}. \tag{66}$$

Since $f_0 = f(\phi_0(t))$ and V_0 may evolve in time, both $\Lambda_4^{(\mathrm{eff})}$ and $G_4^{(\mathrm{eff})}$ are generally time-dependent, unless the scalar field freezes. Note that these quantities are instantaneous —evaluated at the current on-brane value ϕ_0—and reduce to the usual constants only if ϕ_0 is constant. Neglecting the $\mathcal{O}(\rho^2)$ high-energy corrections and the dark radiation term C/a_0^4, the Friedmann equation (39) reduces to

$$H_0^2 = \frac{\Lambda_4^{(\mathrm{eff})}}{3} + \frac{8\pi G_4^{(\mathrm{eff})}}{3}\rho - \frac{k}{a_0^2}. \tag{67}$$

This is precisely the form of the standard 4D FRW equation, but with effective gravitational and cosmological constants determined by the higher-dimensional parameters κ_5^2, λ, Λ_5, V_0, and f_0.

4.2.2 Late-time acceleration analysis

The condition for an accelerating universe is $\ddot{a}/a > 0$. Considering Eq. (40), the necessary and sufficient condition for acceleration is

$$-\frac{\kappa_5^4}{36f_0^2}(\rho_b+\lambda)(\rho_b+3p_b-2\lambda)-\frac{\kappa_5^2}{3f_0}\left(\dot{\phi}_0^2-V_0-\Lambda_5\right)-\frac{C}{a_0^4}\ >\ 0. \tag{68}$$

We can interpret the three contributions as follows:

- **Matter/tension term (junction contribution):** Its effect depends on the sign of $\rho_b+3p_b-2\lambda$; it can either decelerate or accelerate the expansion. In particular, for $\rho_b\to 0$, it gives a positive contribution $\kappa_5^4\lambda^2/(18f_0^2)$, corresponding to vacuum-driven acceleration from the brane tension.
- **Scalar/bulk term:** This term accelerates the brane if the combined potential energy and bulk cosmological constant dominates over the scalar field's kinetic energy:

  $$V_0+\Lambda_5>\dot{\phi}_0^2.$$

  As a result, at low energies, the scalar potential together with Λ_5 effectively behaves as dark energy.
- **Dark radiation term ($\mathcal{C}/a^4$):** Always decelerates the expansion if $\mathcal{C}>0$.

In the special case where the brane contains only tension (negligible matter at late times), one can set $\rho_b\to 0$, $p_b\to 0$, $k=0$, and assume $\mathcal{C}$ negligible. Then (40) reduces to

$$\frac{\ddot{a}}{a}=\frac{\kappa_5^4}{18f_0^2}\lambda^2-\frac{\kappa_5^2}{3f_0}\left(\dot{\phi}_0^2-V_0-\Lambda_5\right). \tag{69}$$

Hence, late-time acceleration ($\ddot{a}/a>0$) requires

$$\frac{\kappa_5^4}{18f_0^2}\lambda^2>\frac{\kappa_5^2}{3f_0}\left(\dot{\phi}_0^2-V_0-\Lambda_5\right), \tag{70}$$

or equivalently,

$$V_0+\Lambda_5>\dot{\phi}_0^2-\frac{\kappa_5^2\lambda^2}{6f_0}.$$

If the scalar has settled so that $\dot{\phi}_0\approx 0$, the condition simplifies to

$$V_0+\Lambda_5>-\frac{\kappa_5^2\lambda^2}{6f_0},$$

which is typically satisfied unless the right-hand side is large and negative.

5 Conclusion

In this work, we constructed a braneworld model that incorporates a scalar field non-minimally coupled to gravity via the term $f(\phi)\mathcal{R}$ within the Palatini formalism, in which the metric and connection are treated as independent variables. The independent variation leads to a conformally related metric connection and modified bulk and brane field equations, as well as a modified equation of motion for the scalar field. We derived generalized junction conditions that allow consistent embedding of the brane with matter and scalar fields. Interestingly, the Palatini coupling changes the structure of the scalar boundary conditions: even with a constant brane tension and no explicit dependence of standard matter fields on ϕ, the geometry (extrinsic curvature jump) sources a jump $[\phi']$ unless $f_{,\phi}(\phi_0) = 0$.

We obtained the modified Friedmann and Raychaudhuri equations and analyzed the scalar dynamics on the brane, which are distinctly different from the metric formulation. These modifications introduce new features into the cosmological evolution at both early and late times, providing a framework to address inflation and late-time cosmic acceleration in a unified manner.

In the high-energy, scalar-dominated regime, the Palatini corrections modify the effective slope $S(\phi)$ governing the scalar dynamics, leading to altered slow-roll parameters ϵ and η, and the total number of e-folds N. Both small ($|\alpha|\phi^2 \ll 1$) and strong ($|\alpha|\phi^2 \gg 1$) non-minimal couplings $f(\phi) = 1 + \alpha\phi^2$ were analyzed, showing that negative α flattens the slope and prolongs inflation, while positive α steepens it. We found that the bulk cosmological constant Λ_5 further modulates slow-roll, enhancing inflation if negative and suppressing it if positive. In addition, modifications to the slow-roll parameters consequently alter the main perturbation parameters—the scalar and tensor spectral indices. This can ultimately determine the model's viability in light of recent observational data. To test the model's viability, we constrained its parameter space using the Planck2018 TT, TE, EE+lowE+lensing+BAO+BK14 data and Planck2018 TT, TE, EE+lowE+lensing+BAO+BK18 data to obtain the observationally viable ranges for the model's parameter α. Our analysis shows that the non-minimal Palatini brane-world formulation provides a natural mechanism for reconciling quadratic-type potentials with observational data. Although the quadratic potential in the standard single-field scenario is observationally disfavored [32], the same potential within our non-minimal Palatini framework remains consistent with data.

At late times, in the low-energy regime ($\rho_b \ll \lambda$) with negligible dark radiation, the system approaches a scalar-dominated or effective vacuum de Sitter solution. The effective 4D Newton constant $G_4^{(\text{eff})}$ and cosmological constant $\Lambda_4^{(\text{eff})}$ were identified, showing that brane tension, scalar potential, and bulk contributions jointly determine the late-time dynamics. Importantly, the interplay between the effective Newton constant and brane tension realizes a generalized Randall-Sundrum fine-tuning condition, ensuring a small effective cosmological constant on the brane while maintaining consistent gravitational coupling. Late-time acceleration is generically achieved when the scalar kinetic energy is subdominant and the scalar potential plus bulk cosmological constant dominate. Furthermore, we showed that energy exchange between the brane fluid and the bulk scalar alters the redshifting of matter, leaving radiation unaffected but modifying dust dilution in a coupling–dependent way.

Overall, our analysis demonstrates that the Palatini brane-world setup with a bulk scalar provides a consistent and unified framework for both early-universe inflation and late-time acceleration. The dynamics are controlled by the interplay of brane tension, scalar potential, non-minimal coupling, and bulk cosmological constant, while the Palatini formalism introduces distinctive corrections to the Friedmann equations and scalar dynamics.

Authors' Contributions

The authors contributed to data analysis, drafting, and revising of the paper and agreed to be responsible for all aspects of this work.

Data Availability

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or non profit sectors.

References

[1] Sotiriou, T. P., & Faraoni, V. 2010, Reviews of Modern Physics, 82, 451.

[2] Olmo, G. J. 2011, International Journal of Modern Physics D, 20, 413.

[3] Bauer, F., & Demir, D. A. 2008, Phys. Lett. B, 665, 222.

[4] Tenkanen, T. 2017, JCAP, 12, 001.

[5] Koivisto, T. 2006, Classical and Quantum Gravity, 23, 4289.

[6] Callan, C. G., Coleman, S. R., & Jackiw, R. 1970, Annals of Physics, 59, 42.

[7] Birrell, N. D., & Davies, P. C. W. 1982, Quantum Fields in Curved Space, Cambridge University Press.

[8] Faraoni, V. 1996, Phys. Rev. D, 53, 6813.

[9] Maeda, K.-i. 1989, Phys. Rev. D, 39, 3159.

[10] Salopek, D. S., & Bond, J. R. 1990, Phys. Rev. D, 42, 3936.

[11] Kaiser, D. I. 1995, Phys. Rev. D, 52, 4295.

[12] Boisseau, B., Esposito-Farèse, G., Polarski, D., & Starobinsky, A. A. 2000, Physical Review Letters, 85, 2236.

[13] Hwang, J.-c., & Noh, H. 2002, Phys. Rev. D, 66, 084009.

[14] Goldberger, W. D., & Wise, M. B. 1999, Physical Review Letters, 83, 4922.

[15] Dvali, G., & Tye, S. H. H. 1999, Phys. Lett. B, 450, 72.

[16] DeWolfe, O., Freedman, D. Z., Gubser, S. S., & Karch, A. 2000, Phys. Rev. D, 62, 046008.

[17] Kanti, P., Kogan, I. I., Olive, K. A., & Pospelov, M. 1999, Phys. Lett. B, 468, 31.

[18] Kanno, S., & Soda, J. 2002, Phys. Rev. D, 66, 083506.

[19] Bogdanos, C., Dimitriadis, A., & Tamvakis, K. 2007, Class. Quant. Grav., 24, 3701.

[20] Bogdanos, C., & Tamvakis, K. 2007, Phys. Lett. B, 646, 39.

[21] Randall, L., & Sundrum, R. 1999, Physical Review Letters, 83, 3370.

[22] Randall, L., & Sundrum, R. 1999, Physical Review Letters, 83, 4690.

[23] Binétruy, P., Deffayet, C., & Langlois, D. 2000, Nuclear Physics B, 565, 269.

[24] Shiromizu, T., Maeda, K.-i., & Sasaki, M. 2000, Phys. Rev. D, 62, 024012.

[25] Maartens, R., Wands, D., Bassett, B. A., & Heard, I. P. C. 2000, Phys. Rev. D, 62, 041301.

[26] Davis, S. C. 2002, JHEP, 03, 058.

[27] Langlois, D., & Rodriguez-Martinez, M. 2001, Phys. Rev. D, 64, 123507.

[28] Bazeia, D., et al. 2015, Eur. Phys. J. C, 75, 569.

[29] Gu, B.-M., Guo, B., Yu, H., & Liu, Y.-X. 2015, Phys. Rev. D, 92, 024011.

[30] da Silva, P. M. L. T., & da Silva, J. M. H. 2016. [arXiv:1603.04793 [hep-th]]

[31] Gu, B.-M., Liu, Y.-X., & Zhong, Y. 2018, Phys. Rev. D, 98, 024027.

[32] Akrami, Y., et al. 2020, Astron. Astrophys., 641, A10.

[33] Paoletti, D., Finelli, F., Valiviita, J., & Hazumi, M. 2022, Phys. Rev. D, 106, 083528.