Iranian Journal of Astronomy and Astrophysics
Vol. 12, No. 1, Winter 2025, 7–36
©Available online at http://ijaa.du.ac.ir
DOI: 10.22128/ijaa.2025.965.1207
Online ISSN: 2383–403X

Iranian Journal of Astronomy and Astrophysics

Research Paper

Dirac Stars Stability and Modified Einstein-Dirac-Maxwell Gravity

Hossein Ghaffarnejad*[1] · Tohid Ghorbani[2]

[1] Faculty of Physics, Semnan University, P.C. 35131–19111, Semnan, Iran;
*email: hghafarnejad@semnan.ac.ir

[2] Faculty of Physics, Semnan University, P.C. 35131–19111, Semnan, Iran;
email: tohidghorbani@semnan.ac.ir

Received: 22 February 2025; **Accepted:** 24 May 2025; **Published:** 27 July 2025

Abstract. To consider the cosmic magnetic effects on the rate of cosmic inflation, instead of unknown dark sector of matter/energy, some authors presented non-minimally coupled exotic Einstein-Maxwell (EM) gravity theories which we address in this work. We use one of these models and add Dirac action functional interacting with gauge Maxwell field to the exotic EM gravity and then investigate the formation and stability of a relativistic fermion star by using the dynamical system approach. Mathematical calculations predict important role of frequencies of the Dirac waves in formation of the Dirac star and whose critical energy density. Dirac star is a particular kind of fermionic relativistic star which has spherically symmetric static metric field. In this context, the directional interaction parameter between the gravity, and the electromagnetic fields play more important role, particularly in size of the star. Large values of that parameter makes larger Dirac star. Furthermore, we apply the dynamical system approach to find stabilization conditions of the Dirac star. These conditions are linked to specific values of the total angular momentum quantum numbers (including both spin and orbital contributions).

Keywords: Relativistic Stars, Fermionic, Dirac Spinor, Stability, S-Mode Wavs, Dynamical Systems.

1 Introduction

To describe the expansion of the universe via particle physics perspective where gravity plays a dominant role, usually the standard model is generalized by the general theory of relativity as gravity side of the system. Such models are called scalar-vector-tensor-spinor gravity theories, where the combination of spinor fields with the electromagnetic field and metric tensor fields play a crucial role in achieving appropriate models that describe expanding universe [1–3]. Such models are also used to describe how compact stellar objects form. See for instance [4,5] for formation of star and wormhole respectively with baryonic matter. For application of the Yang-Mills theory coupled with the Einstein metric field equations, one can see [6]. Finster et al. used numerically method in ref [7], to solve Einstein-Dirac (ED) equations and found regular and well-behaved metric solutions which describe a neutral spherically symmetric coupled static compact stellar system with two

*Corresponding author

singlet spin-1/2 fermions. They used Einstein-Dirac-Maxwell (EDM) equations to study the effects of gauge field for star formation the references [8]. [9,10] are similar works but by using the perturbation method. Solitonic particle-like stable solutions of the EDM gravity called as Dirac stars (DS) are found in presence of spinor self-interaction [11–22]. Similar research has been done to achieve the concept of boson stars (BS) [23]. Both BS and DS are proposed as approximate descriptive models for microscopic objects in the early universe [24]. Investigations in the classical approach of field theories predict that there are no different properties between DS and BS [4,25], but in the quantum approach, a rotating BS can have a black hole horizon inside it, unlike a DS [4]. Moreover, macroscopic BSs are believed to exist in nature while microscopic DSs are considered purely mathematical abstractions. The stability of the spinor field has been investigated in excited states [4,7,8,21,26,27], in addition to the ground state, as well as in the states coupled with the scalar field to form Dirac-boson stars (DBSs) [28].
Coupling of the ED field with Maxwell field to form charged DS can lead to more stable solutions [8]. In this manner, researchers found solutions for the ground state of the rotating charged DS [24,29].
Some other applications of the EDM gravity in presences of interaction potentials can be followed in Refs. [22,30–37]. In all of studies above the field equations are nonlinear usually and hence the methodologies fall into two categories are two different kinds: numerical studies or perturbation analytical studies. To study stability situations of the obtained solutions one usually uses systems approach (See for instance [38] or Introduction section of Ref. [39] and as an application in the stability of boson star see [40]). In this approach, one find linear order perturbation solutions of the fields near the critical points. Methodology of this approach has 5 steps:

a All higher order derivatives of the field equations are called with new fields such that all equations reach to first order differential equations in phase space. Dimensions of such a phase space are number of first order derivative of the fields which we can consider that they are components of a vector field in assumed phase space. In that case in vector field representation we can write

$$\dot{\vec{\Lambda}} = \frac{d\vec{\Lambda}}{dt} = \vec{F}(\vec{\Lambda}, t), \tag{1}$$

in the phase space. If $\vec{F}$ is independent of the parameter t which we assume here is the time then, the equation (1) is called as autonomous but if not be, then it is called as non-autonomous. Usually non-autonomous kind reaches to chaotic systems where stable critical points are attractor or absorber chaotic system [41].

b The second step is determination of the critical points which are obtained by solving the equation $\dot{\vec{\Lambda}} = 0$.

c The third step is linear making of the equation (1). This is done by calculating the Jacobi matrix

$$J_i^j = \frac{\partial F_i}{\partial \Lambda_j}\bigg|_{critical\ point} \tag{2}$$

at the critical points.

d After to have numeric form of the Jacobi matrix (2) then, we apply to solve the following set of linear order differential equations instead of (1).

$$\dot{\Lambda}_i = \frac{d\Lambda_i}{dt} = \sum_{j=1}^{n} J_i^j \Lambda_j. \tag{3}$$

e To investigate which of the obtained linear order solutions of the equation (3) are stable, we must determine sign of eigenvalues s of the Jacobi matrix (2) by solving its secular equation $\det\{J_i^j - s\delta_i^j\} = 0$. There are two different cases for numeric values of the eigenvalues. If they are real, then the system is stable, if all real roots have negative sign, but the system is quasi stable, if some of roots have negative sign and some others have positive sign. In the case, where the roots of the secular equation give us some complex numeric values for the eigenvalues, then, the system will be spiral stable (see figure 1(d)) if real part of these complex numeric eigenvalues be negative but the system will be quasi-spiral stable when some of real parts of the complex eigenvalues have negative and some other have positive sign. The system under consideration will be unstable if all real roots of secular equation have positive sign or when they are complex numeric, then all real parts of them have positive sign.

Our motivation in this work is to use the dynamic system approach above to investigate stability conditions of a Dirac star defined by a non-minimal interacting EDM gravity model [42]. Importance of such a gravity model is outlined below, which was applied previously to describe cosmos. Layout of the paper is as follows:
In section 2, we define the gravity model, and give out the equations of the fields. Also we define particular choice of the Dirac matrices which make hermitian the Dirac Hamilton operator. This is very important to separate radial and angular parts of the Dirac fields versus the spherical harmonic eigenfunctions. In sections 3, we set all Einstein, Maxwell and the Dirac field equations for a spherically symmetric static metric background. In subsection 3.1, we notify the charge conjugation operator. In subsection 3.2, we give out radial part of the Dirac fields. Section 4 denotes to nonzero components of stress tensor of the fields and use them to generate field equations of the system under consideration in Section 5. In the latter section we find some suitable formulas for the energy density and directional pressure of the Dirac star. Also we make eight-dimensional phase space of the system with eight first order differential equations by according the description which is given in the introduction section for the dynamical system approach. Also we obtain Jacobi matrix, critical points and eigenvalues of the Jacobi matrix. In subsection 5.1, we solve just one choice of four kind possible solutions of the Dirac equation, i.e., the spinor have up spin. In Subsection 5.2, we estimate the radius of the obtained Dirac star versus the interaction coupling constant of the action functional. The final section of the paper is dedicated to concluding remarks and outlook of the work.

2 The gravity model

Let us call the exotic modified EM gravity given by [42] and add the Dirac spinor action functional in which total action is now

$$I = \int d^4x\sqrt{g}\left[\frac{R}{16\pi} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \alpha F_{\rho\mu}R^{\mu}{}_{\eta}F^{\eta\rho} + \beta(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - m_{\psi}\bar{\psi}\psi) + \eta A_{\mu}j^{\mu}\right]. \tag{4}$$

With $\beta = 0 = \eta$, this model proposed firstly by Turner to investigate cosmic magnetic field effects on cosmic inflation where conformal symmetry breaking and gauge symmetry breaking cause to be dominant the cosmic magnetic field energy density versus the usual vacuum

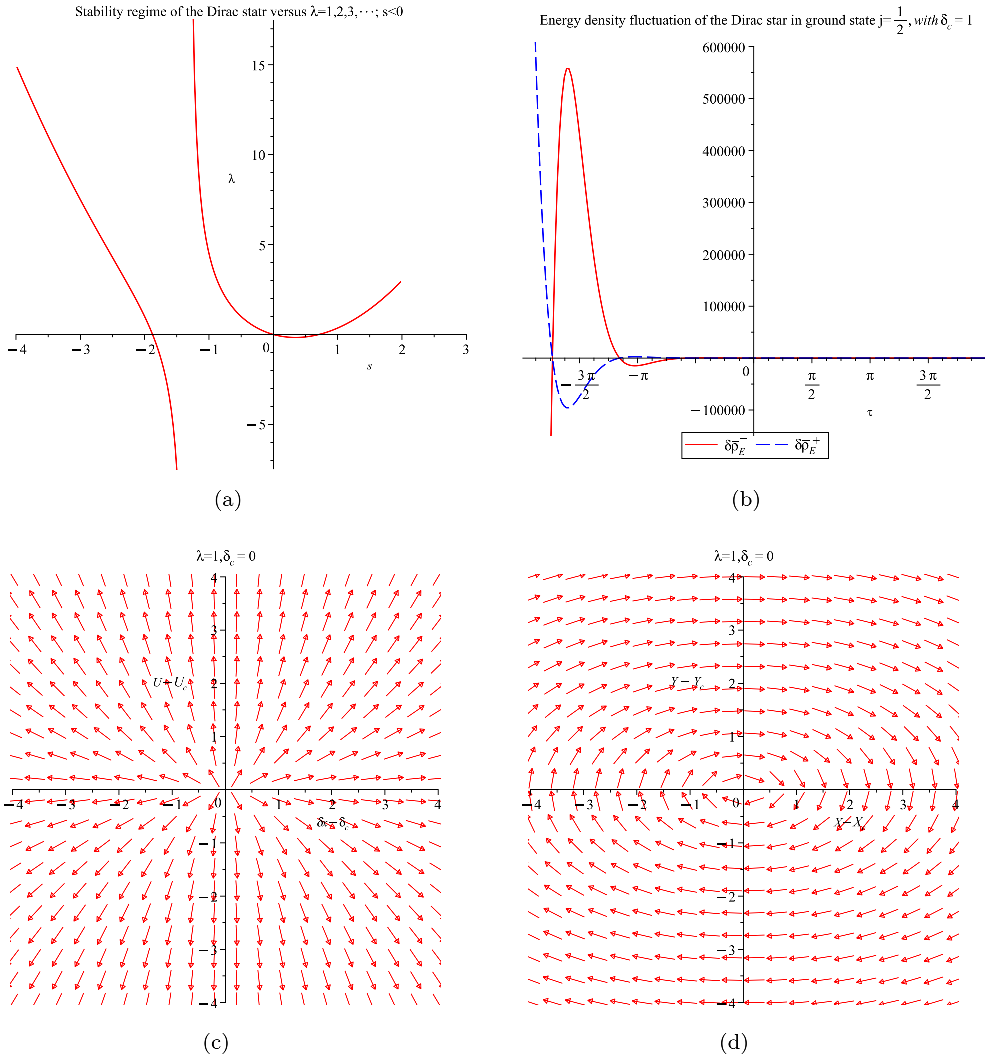

Figure 1: (a) Permissible values for eigenvalues in stable state of the Dirac star. Negative values of the eigenvalues $s < 0$ for each of quantized states $\lambda = 1, 2, 3, 4 \cdots$ describe stable state of the system , (b) Variation of energy density difference vs radial coordinate. There are two different branches of the solutions which describe decreasing density which diverge to infinite value at center $\tau \to -\infty$ of the star. (c) Source (unstable) nature in phase space. (d) Sink (stable) nature in phase space. The arrow diagrams show that the system is in quasi stable state.

(unknown) dark energy density. This comes from the exotic non-minimal directional interaction α term. While the ordinary Einstein Maxwell gauge invariant gravity theory is not a suitable model to do so. According to this motivation, we encouraged to use this model to study the effects of exotic directional coupling interaction part between gravity and the electromagnetic field in formation and stability of a fermionic star in this work.
In the above action functional the interaction parameter between the gravity and the electromagnetic fields α has dimension of $(length)^2$ in the geometric units $c = G = 1$, and other interaction parameters β and η between the Dirac fermions and the gravity and the electromagnetic fields are dimensionless.. The mass of Dirac particles m_ψ has an inverse length dimension and the vector potential is dimensionless field. The Dirac spinors have dimensions of $(length)^{-1/2}$. In the above action the electric current density j^μ defined in terms of the Direct versus the Dirac spinor ψ and corresponding adjoint spinor

$$\bar{\psi} = \psi^\dagger(-i\tilde{\gamma}^t), \tag{5}$$

and its electric charge q_e such that

$$j^\mu = iq_e\bar{\psi}\gamma^\mu\psi, \tag{6}$$

where we use the convention of reference [43] to define the adjoint spinor form above, but in more common usage, the imaginary unit 'i' is suppressed by changing the Dirac γ-matrices as $\gamma^\mu \to i\gamma^\mu$. In the latter case, we must use the convention given in [43] again, for representation of the Dirac γ-matrices which in the flat Minkowski spacetime with a Cartesian coordinates system are

$$\tilde{\gamma}^0 = i\begin{pmatrix} \mathrm{I} & \mathtt{0} \\ \mathtt{0} & -\mathrm{I} \end{pmatrix} = i\sigma^3 \otimes \mathtt{I}, \quad \tilde{\gamma}^j = i\begin{pmatrix} \mathtt{0} & \sigma^j \\ -\sigma^j & \mathtt{0} \end{pmatrix} = -\sigma^j \otimes \sigma^2, \tag{7}$$

in which

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathtt{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

and $j \equiv 1, 2, 3$ or x, y, z respectively and σ^j are Pauli matrices such that

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

It is obvious that the difference between (7) and γ- matrices represented in field theory books is only the factor of 'i'. As we see below, usage of the above representation makes that the Dirac Hamiltonian operator to be hermitian and so will be commutes with others such as angular momentum and spin-orbit coupling operator. This is very important to make separable the radial part with angular part of the Dirac field in the spherically symmetric space time. As we see at below that the angular part of the Dirac field by regarding the choice (7) will be can be expressed in terms of the spherical harmonic eigenfunctions. In the above action functional, g is the absolute value of the determinant of the metric field $g_{\mu\nu}$. The non-minimal susceptibility tensor [44] and in extended version can be defined by the Reimann and Weyl tensors. $R_{\mu\nu}(R)$ is Ricci tensor (scalar). The β parameter is coupling constant between the Dirac field and the gravity $g_{\mu\nu}$. γ^μ is Dirac γ-matrix in curved spacetime. m_ψ is mass parameter for the Dirac particles. The η parameter is coupling constant between the electromagnetic vector potential A_μ and the four electric current density j^μ. Substituting the Dirac equation of motion (25) one can show that j^μ satisfies the covariant conservation condition $\nabla_\mu j^\mu = 0$. Hence by keeping $\beta = \eta$ it is convenient to move the last η term of the action functional (4) into the covariant derivative ∇_μ such that

$$D_\mu = \nabla_\mu + iq_e A_\mu = \partial_\mu + iq_e A_\mu - \Gamma_\mu, \tag{10}$$

in which Γ_μ is spinor connection [45–48]

$$\Gamma_\mu = -\frac{1}{4}\tilde{\gamma}^a\tilde{\gamma}^b\omega_{ab\mu}, \tag{11}$$

and the spin connection $\omega_{ab\mu}$ is defined by

$$\begin{aligned} \omega_{ab\mu} &= \eta_{ac}\omega^c{}_{b\mu} = e_{\alpha\beta}\nabla_\mu e_b{}^\beta = g_{\beta\alpha}e_\alpha{}^\beta\nabla_\mu e_b{}^\beta, \\ \omega^a{}_{b\mu} &= -e_b{}^\nu(\partial_\mu e^a{}_\nu - \Gamma^\lambda{}_{\mu\nu}e^a{}_\lambda). \end{aligned} \tag{12}$$

The spin connection is antisymmetric with respect to the two Minkowski indices a and b, i.e.,

$$\omega_{ab\mu} = -\omega_{ba\mu}, \quad \omega_{aa\mu} = 0. \tag{13}$$

The vierbein fields or tetrads $e_a{}^\mu$ relate the metric tensor field in Minkowski space-time to the metric tensor field in curved space-time, such that [49]

$$\eta_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}, \quad g_{\mu\nu} = e^a{}_\mu e^b{}_\nu\eta_{ab}, \tag{14}$$

where, the indices a and b correspond to Minkowski space-time, and the indices μ and ν correspond to curved space-time. By considering D_μ instead of ∇_μ the electromagnetic part of the action functional (4) remains unchanged because

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{15}$$

where the last equality is valid just for torsion free Riemannian geometries. Thus we do not consider the last η term in the action functional (4), when we calculate equation of motion of the fields in what follows but we must regard D_μ instead of ∇_μ in the Dirac equations of motion. In the above equation $\tilde{\gamma}^a$ is γ-matrix in the flat Minkowski space $\eta_{ab} = diag(-1, +1, +1, +1)$ defined in the Cartesian coordinates, and it relates to the γ-matrix in curved space through the vierbein

$$g_{\mu\nu} = e_{a\mu}e^a{}_\nu, \quad \eta_{ab} = e_{a\mu}e_b{}^\mu, \tag{16}$$

such that

$$\gamma^\mu = e_a{}^\mu\tilde{\gamma}^a, \tag{17}$$

and so

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab}, \tag{18}$$

where $\{x, y\} = xy + yx$ is the anti-commutator. Varying the action functional (4) with respect to the metric tensor field $g^{\mu\nu}$, the Einstein metric field equation is found as follows.

$$G_{\mu\nu} = 8\pi T^{total}_{\mu\nu} = 8\pi(T^{EM}_{\mu\nu} + \alpha\Theta_{\mu\nu} + \beta T^{Dirac}_{\mu\nu}), \tag{19}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{20}$$

is the Einstein tensor defined by the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$,

$$T^{EM}_{\mu\nu} = -\frac{1}{8}\left[F_{\mu\alpha}F^{\alpha}_{\nu} + F_{\beta\nu}F^{\beta}_{\mu} - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right], \tag{21}$$

is traceless electromagnetic Maxwell fields stress tensor,

$$\begin{aligned}
&\Theta_{\mu\nu} = \Phi_{\mu\nu} + \nabla_{\eta}\Omega^{\eta}_{\mu\nu} - \frac{\nabla_{\mu}\Omega^{\eta}_{\eta\nu}}{2} + \frac{\nabla_{\nu}\Pi_{\mu}}{2} + \frac{g_{\mu\nu}\Delta}{2},\\
&\Phi_{\mu\nu} = F_{\nu\xi}R^{\xi\eta}F_{\eta\mu},\\
&\Omega^{\eta}_{\mu\nu} = \frac{\nabla_{\mu}(\sqrt{g}\mathcal{O}^{\eta}_{\nu})}{\sqrt{g}},\\
&\mathcal{O}^{\eta}_{\nu} = F^{\eta\lambda}F_{\nu\lambda},\\
&\Pi_{\mu} = \Omega^{\eta}_{\eta\mu} - \Omega^{\eta}_{\mu\eta},\\
&\Delta = \Phi^{\mu}_{\mu} - \nabla^{\xi}\Omega^{\eta}_{\eta\xi},
\end{aligned} \tag{22}$$

is gravity-photon interaction stress tensor and

$$T^{Dirac}_{\mu\nu} = -\frac{1}{4}[\bar{\psi}\gamma_{\mu}D_{\nu}\psi + \bar{\psi}\gamma_{\nu}D_{\mu}\psi - (D_{\mu}\bar{\psi})\gamma_{\nu}\psi - (D_{\nu}\bar{\psi})\gamma_{\mu}\psi], \tag{23}$$

is the Dirac spinor matter field stress tensor in which

$$D_{\mu}\psi = \partial_{\mu}\psi + (iq_eA_{\mu} - \Gamma_{\mu})\psi, \quad D_{\mu}\bar{\psi} = \partial_{\mu}\bar{\psi} + (iq_eA_{\mu} + \Gamma_{\mu})\bar{\psi}. \tag{24}$$

Also, by varying the action functional (4) with respect to the adjoint Dirac field $\bar{\psi}$ and the Electromagnetic Maxwell tensor field $F_{\mu\nu}$ we obtain corresponding equations of motion respectively such that

$$\gamma^{\mu}D_{\mu}\psi - m_{\psi}\psi = 0, \tag{25}$$

and

$$\nabla_{\nu}\tilde{F}^{\mu\nu} = 2\beta j^{\mu}, \tag{26}$$

where we define modified anti-symmetric Maxwell tensor field as

$$\tilde{F}^{\mu\nu} = F^{\mu\nu} - 2\alpha\left(R^{\nu}_{\eta}F^{\eta\mu} - R^{\mu}_{\eta}F^{\eta\nu}\right), \tag{27}$$

and for arbitrary anti-symmetric tensor $O^{\mu\nu}$ defined in torsion free curved spacetimes, we have

$$\nabla_{\mu}O^{\mu\nu} = \frac{\partial_{\mu}(\sqrt{g}O^{\mu\nu})}{\sqrt{g}}. \tag{28}$$

After presenting the gravity model under consideration, we set the dynamical equations for a spherically symmetric static curved line element in the subsequent section.

3 Spherical Dirac spinors

We consider a general form of spherically symmetric static curved spacetime whose line element is given by

$$ds^2 = -e^{2U(r)}dt^2 + e^{2V(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{29}$$

where the spacetime can be foliated by time-independent spatial hypersurfaces Σ_t. Induced metric γ_{ij} on Σ_t and the future directed, timelike normal unit vector field n^μ to Σ_t are [50]

$$\gamma_{ij} = diag(e^{2V(r)}, r^2, r^2\sin^2\theta), \quad n^\mu = (e^{-U(r)}, 0, 0, 0). \tag{30}$$

Because of intrinsic spin property of a single Dirac fermion, the spherically symmetric property cannot be preserved in the Dirac spinors ψ. In other words, since the spin of a fermion has an intrinsic orientation in space, a system consisting of a single Dirac particle cannot be spherically symmetric. To have spherical spinors two different approaches are presented in the scientific literature (see [43] and references therein). The first (old) approach is based on the first quantization approach, i.e., the relativistic quantum mechanics which is in accord with the Finster et al works where the authors consider two fermions having opposite spin (a singlet spinor state) [51]. To realize the latter picture (twin up/down fermion model), they considered the electromagnetic interaction with them in ref. [52]. To see other extensions of this approach, i.e., time dependent curved spacetimes and so on, one can follow papers given by [43], but the second approach considers the quantum field theory perspective (the second quantization) which is applicable even if we have a single Dirac field. This is presented by Ben Kain who is author of the work [43]. He preserved spherical symmetry by focusing on excitations of the vacuum with zeroth total angular momentum. In his construction, static spherically symmetric self-gravitating configurations of spin-1/2 particles in quantum field theory include the presence and expectation value of stress tensor operator for a population of identical quantum particles treated as a fermionic matter source in the Einstein metric equation. In our work, we adopt this latter approach our studies about stability condition of a fermion-electromagnetic-spherically symmetric static stellar object. To obtain exact form of the Dirac equation (25) for the line element (29) for which $\psi(t, r, \theta, \varphi)$ could be separable into a multiplication of four functions with 'one' variable, we must first obtain explicit forms of the Dirac matrices (17), spin connections (12) and spinor connections (11) and other quantities versus the spherical coordinate system in curved spacetime (29). It is easy to show that the Dirac matrices (7) can be represented in the spherical polar coordinates system of a Minkowski space-time by the following polar transformations

$$\begin{aligned}
\tilde{\gamma}^t &= \tilde{\gamma}^0, \\
\tilde{\gamma}^r &= \tilde{\gamma}^1 \sin\theta\cos\varphi + \tilde{\gamma}^2 \sin\theta\sin\varphi + \tilde{\gamma}^3\cos\theta, \\
\tilde{\gamma}^\theta &= \tilde{\gamma}^1 \cos\theta\cos\varphi + \tilde{\gamma}^2 \cos\theta\sin\varphi - \tilde{\gamma}^3\sin\theta, \\
\tilde{\gamma}^\varphi &= -\tilde{\gamma}^1 \sin\varphi + \tilde{\gamma}^2\cos\varphi.
\end{aligned} \tag{31}$$

At last, to have their representations in curved spacetime (29) we need tetrads or vierbein fields of the line element (29) defined by (14). With $\eta_{ab} = diag(-1, +1, +1, +1)$ defined in the Cartesian coordinates, the identities (14) for the line element (29) reduce to the following vierbein field components.

$$\begin{aligned}
e^\mu_a &= \left(e^{-U(r)},\ e^{-V(r)},\ \frac{1}{r},\ \frac{1}{r\sin\theta}\right), \\
e^a_\mu &= (e^{U(r)},\ e^{V(r)},\ r,\ r\sin\theta),
\end{aligned} \tag{32}$$

where $a \equiv \{0, 1, 2, 3\} \equiv \{t, x, y, z\}$ and $\mu \equiv \{t, r, \theta, \varphi\}$. Substituting (32) into the vierbein transformation (17) we obtain

$$\gamma^t = e^{-U(r)}\tilde{\gamma}^t, \quad \gamma^r = e^{-V(r)}\tilde{\gamma}^r, \quad \gamma^\theta = \frac{\tilde{\gamma}^\theta}{r}, \quad \gamma^\varphi = \frac{\tilde{\gamma}^\varphi}{r\sin\theta}. \tag{33}$$

Using the explicit form of the Vierbein field (32) and relations (12) and (13),and some mathematical calculations to find non-vanishing components of the spin connections (12), one find finally, explicit form of the spinor connections (11) for the line element (29), such that

$$\begin{aligned}\Gamma_t &= \frac{U'}{2}e^{U-V}\tilde{\gamma}^t\tilde{\gamma}^r,\\ \Gamma_r &= 0,\\ \Gamma_\theta &= \frac{1}{2}(1-e^{-V})\tilde{\gamma}^\theta\tilde{\gamma}^r,\\ \Gamma_\varphi &= \frac{1}{2}(1-e^{-V})\sin\theta\tilde{\gamma}^\varphi\tilde{\gamma}^r,\end{aligned}$$

which by using (33) read to the following scalar

$$\gamma^\mu\Gamma_\mu = e^{-V}\left[\frac{e^V-1}{r} - \frac{U'}{2}\right]\tilde{\gamma}^r. \tag{34}$$

Although there is other choice of the Vierbein field (see eq. 21 and Appendix C in ref. [43]) it is not suitable for our goal in this work. That is because with the vierbein field (32), the Dirac equation can be solved by separation of variables method in which we see next, that its angular part is defined by spherical Harmonic Polynomials. By having the above results we are in position to solve the Dirac equation (25) for spherical spinors. This is done in what follows, by using the separable of variables method. To have ortho-normal (normalized orthogonal) mode solutions of the Dirac equation we point that the line element (29) is time-independent having a time like Killing vector field $\xi^\mu = (1,0,0,0)$. The existence of such a vector field generates coordinate independent positive (negative) frequency Dirac mode solutions $\chi_+(\chi_-)$ [45,46,53] such that we have

$$\xi^\mu\partial_\mu\chi_\ell^\mp = \partial_t\chi_\ell^\mp = \pm i\omega\chi_\ell^\mp, \tag{35}$$

for real positive ω and orthogonality condition

$$\begin{aligned}(\chi_j^+,\chi_\ell^+) &= (\chi_j^-,\chi_\ell^-) = \delta_{j\ell},\\ (\chi_j^+,\chi_\ell^-) &= (\chi_j^-,\chi_\ell^+) = 0,\end{aligned} \tag{36}$$

where j and ℓ denote to symbols of different quantum numbers and (,) denotes to inner product of two mode solutions on the spatial hypersurface Σ_t as

$$(\psi_1,\psi_2) = \int_{\Sigma_t} d^3x\sqrt{det(\gamma_{ij})}\psi_1^\dagger\psi_2 = \frac{1}{q_e}\int_{\Sigma_t} d^3x\sqrt{det(\gamma_{ij})}n_\mu j^\mu. \tag{37}$$

In the above relation $j^\mu = iq_e\bar{\psi}_1\gamma^\mu\psi_2$ and $dx^3 = drd\theta d\varphi$ and $\det(\gamma_{ij}) = e^{2V}r^4\sin^2\theta$ with n_μ are given by (30). By regarding the orthogonality conditions (36) we can now express the series expansion form of the Dirac spinors, such that

$$\psi(t,\vec{r}) = \sum_\ell\int d\omega[a_\ell(\omega)\chi_\ell^+(t,\vec{r}\,;\omega) + b_\ell(\omega)\chi_\ell^-(t,\vec{r}\,;\omega)], \tag{38}$$

where the coefficients $a_\ell(\omega)$ and $b_\ell(\omega)$ are complex numbers. When ψ treats as operator in quantum field theory perspective then they should describe creation and anhiliation of quantum particles which is not our goal to consider in this paper (see [43]).

At first step, we generate Schrodinger like form of the Dirac equation (25) for electrostatic case, i.e., $A_\mu = A_0(r)\delta^0_\mu$, by multiplying $\gamma^t = e^{-U}\tilde{\gamma}^0$ from left side and by substituting (10) and (34) such that

$$i\partial_t\psi = \hat{H}\psi, \tag{39}$$

in which the Hamilton operator is

$$\hat{H} = (i\tilde{\gamma}^0)\Big[e^U\hat{G} + [qA_0(r)(i\tilde{\gamma}^0) - m_\psi]I - e^{-V}\left(\frac{e^V - 1}{r} - \frac{U'}{2}\right)\tilde{\gamma}^r\Big], \tag{40}$$

and

$$\hat{G} = e^{-V}\tilde{\gamma}^r\partial_r + \frac{\tilde{\gamma}^\theta\partial_\theta}{r} + \frac{\tilde{\gamma}^\varphi\partial_\varphi}{r\sin\theta} = (e^{-V} - 1)\tilde{\gamma}^r\partial_r + \vec{\gamma}\cdot\vec{\nabla}. \tag{41}$$

We now study more the operator $\vec{\gamma}\cdot\vec{\nabla}$ which by substituting (7) can be written versus the Pauli spinors, such that

$$\vec{\gamma}\cdot\vec{\nabla} = \begin{pmatrix} O & i\vec{\sigma}\cdot\vec{\nabla} \\ -i\vec{\sigma}\cdot\vec{\nabla} & O \end{pmatrix}, \tag{42}$$

in which O is two dimensional zero matrix. From quantum field theory we have relation between the spin operator and Pauli spinors $\vec{\sigma}$ as

$$\hat{\vec{S}} = \frac{\hat{\vec{\sigma}}}{2}, \tag{43}$$

for two component spinors and

$$\hat{S} = \frac{\hat{\Sigma}}{2},$$
$$\hat{\Sigma}_1 = \begin{pmatrix} \sigma^1 & O \\ O & \sigma^1 \end{pmatrix}, \quad \hat{\Sigma}_2 = \begin{pmatrix} \sigma^2 & O \\ O & \sigma^2 \end{pmatrix}, \quad \hat{\Sigma}_3 = \begin{pmatrix} \sigma^3 & O \\ O & \sigma^3 \end{pmatrix}, \tag{44}$$

for four component spinors, which in the Dirac representation of gamma matrices (7) reads

$$\hat{\Sigma}_j = -i\tilde{\gamma}^j\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3, \tag{45}$$

where we use $\hbar = 1 = c$ units. However, with this, we can write $\vec{\sigma}\cdot\vec{\nabla} = 2\vec{S}\cdot\vec{\nabla}$ or in momentum operator format $\vec{\sigma}\cdot\vec{\nabla} = 2i\vec{S}\cdot\vec{p}$. Using the identity $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} = i\vec{\sigma}\cdot\vec{a}\times\vec{b}$ for every arbitrary vectors $\vec{a}$ and $\vec{b}$ the equation $\vec{\sigma}\cdot\vec{\nabla}$ reads

$$\begin{aligned}\vec{\sigma}\cdot\vec{\nabla} = i\vec{\sigma}\cdot\vec{p} &= \frac{i(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p})}{r^2} \\ &= i\frac{(\vec{\sigma}\cdot\vec{r})}{r^2}[\vec{r}\cdot\vec{p} + i\vec{\sigma}\cdot(\vec{r}\times\vec{p})] \\ &= i\frac{(\vec{\sigma}\cdot\vec{r})}{r^2}[\vec{r}\cdot\vec{p} + i\vec{\sigma}\cdot\hat{\vec{L}}],\end{aligned} \tag{46}$$

in which

$$\vec{\sigma}\cdot\hat{\vec{L}} = 2\hat{\vec{S}}\cdot\hat{\vec{L}} = \hat{J}^2 - \hat{L}^2 - \hat{S}^2 = \hat{K}, \tag{47}$$

is the well known spin-orbit angular momentum coupling operator. With this result one can infer that the operator (42) reads

$$\vec{\gamma}\cdot\vec{\nabla} = \left(\partial_r - \frac{\hat{K}}{r}\right)\begin{pmatrix} O & i\vec{\sigma}\cdot\hat{n} \\ -i\vec{\sigma}\cdot\hat{n} & O \end{pmatrix} \equiv \left(\partial_r - \frac{\hat{K}}{r}\right)\tilde{\gamma}\cdot\hat{n}. \tag{48}$$

Substituting the above result into the operator of $\hat{G}$ we can write explicit form of the Hamilton operator (40) as

$$\hat{H} = \left[e^{U-V}\partial_r + \frac{(e^{-V} - 1 - e^{U} - \hat{K})}{r} + e^{-V}\frac{U'}{2}\right]\tilde{\gamma}^r + qA_0(r)(i\tilde{\gamma}^0) - m_\psi I, \tag{49}$$

where we use the identity $\tilde{\gamma}\cdot\hat{n} = \tilde{\gamma}^r$. We clime that the above Hamiltonian operator is Hermitian, i.e., $\hat{H}^\dagger = \hat{H}$, because the following operators are Hermitian.

$$(i\tilde{\gamma}^0)^\dagger = i\tilde{\gamma}^0, \quad (\tilde{\gamma}^j)^\dagger = \tilde{\gamma}^j \quad (\tilde{\gamma}^r)^\dagger = \tilde{\gamma}^r, \quad (\hat{K})^\dagger = \hat{K}, \quad I^\dagger = I, \tag{50}$$

which can be checked easily by using (7). This study shows that the operators $i\partial_t$, $\hat{H}$, $\hat{J}^2$, $\hat{J}_z$ and $\hat{K}$ commute with one another and so they have a simultaneous eigenfunctions, i.e., the tensor-spherical harmonics given by (57). In the latter case we can write

$$\begin{aligned} &i\partial_t\psi = \omega\psi, \\ &\hat{H}\psi = \omega\psi, \\ &\hat{K}\psi = \kappa\psi, \\ &\hat{J}^2\psi = j(j+1)\psi, \\ &\hat{J}_z\psi = m_j\psi, \end{aligned} \tag{51}$$

where

$$\begin{aligned} &s = \frac{1}{2}, \\ &l = 0, 1, 2, 3, \cdots, \\ &j = l + s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \\ &m_j = m_\ell + m_s = -j, -(j-1), -(j-2), \cdots - \frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots j-2, j-1, j \\ &m_s = -\frac{1}{2}, \frac{1}{2} \\ &m_\ell = -\ell, -\ell+1, \cdots - 2, -1, 0, 1, 2, \cdots \ell \end{aligned} \tag{52}$$

and κ is eigenvalues of the operator $\hat{K}$ such that

$$\kappa_\pm(j,\ell) = j\left(j + \frac{1}{2}\right) - \ell\left(\ell + \frac{1}{2}\right) - \frac{1}{2}\left(1 + \frac{1}{2}\right), \tag{53}$$

are corresponding eigenvalues and K_- and K_+ denote to two different coupling states between the spin $\vec{\hat{S}}$ and the orbital angular momentum $\vec{\hat{L}}$ respectively as $j = \ell - \frac{1}{2}$ and $j = \ell + \frac{1}{2}$ respectively with exact quantum values

$$\kappa_- = -j - \frac{3}{2} = -(1+\lambda); \quad \ell = j + \frac{1}{2}$$

$$\kappa_+ = j - \frac{1}{2} = +(\lambda - 1); \quad \ell = j - \frac{1}{2}. \tag{54}$$

In the above quantum numbers we have $\lambda = j + \frac{1}{2} = 1, 2, 3, \cdots$. Defining the Dirac spinor versus separable of variables two components spinors as

$$\psi(t, r, \theta, \varphi) = \begin{pmatrix} \phi_1(r, \theta, \varphi) \\ \phi_2(r, \theta, \varphi) \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} \xi(r)A(\theta, \varphi) \\ \zeta(r)B(\theta, \varphi) \end{pmatrix} e^{-i\omega t}, \tag{55}$$

$$\hat{K} \begin{pmatrix} A(\theta, \varphi) \\ B(\theta, \varphi) \end{pmatrix} = \kappa_\pm(j, \ell) \begin{pmatrix} A(\theta, \varphi) \\ B(\theta, \varphi) \end{pmatrix}, \tag{56}$$

where $A(\theta, \varphi)$ and $B(\theta, \varphi)$ can take on both of these coupling states and so we can write them versus the spherical Harmonic eigenfunctions as

$$\mathcal{Y}^{m_j}_{j=\ell\mp\frac{1}{2}}(\theta, \varphi) \equiv \mathcal{Y}^{j=\ell\pm\frac{1}{2}}_\ell(\theta, \varphi) = \pm\sqrt{\frac{\ell \pm m_\ell + \frac{1}{2}}{2\ell + 1}} Y^{m_\ell - \frac{1}{2}}_\ell(\theta, \varphi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{\ell \mp m_\ell + \frac{1}{2}}{2\ell + 1}} Y^{m_\ell + \frac{1}{2}}_\ell(\theta, \varphi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{57}$$

Thus, for two dimensional spinors $\phi_{1,2}$ we must choose two different states such that

$$\phi_{1\pm} = \xi_\pm(r)\mathcal{Y}^{j\pm\frac{1}{2}}_\ell(\theta, \varphi), \quad \phi_{2\pm} = \zeta_\pm(r)\mathcal{Y}^{j\pm\frac{1}{2}}_\ell(\theta, \varphi), \tag{58}$$

and so four dimensional Dirac spinors will be

$$\psi^+_- = \begin{pmatrix} \phi_{1+} \\ \phi_{2-} \end{pmatrix}, \quad \psi^-_+ = \begin{pmatrix} \phi_{1-} \\ \phi_{2+} \end{pmatrix}. \tag{59}$$

Furthermore, the operator

$$\tilde{\sigma}^r = \vec{\tilde{\sigma}} \cdot \hat{n} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}, \quad \hat{n} = \frac{\vec{r}}{r}, \tag{60}$$

commute with the operator $\hat{K}$ (see [43]) and by according to proof which is given in [54] we have

$$\tilde{\sigma}^r \mathcal{Y}^{j\pm\frac{1}{2}}_\ell = -\mathcal{Y}^{j\mp\frac{1}{2}}_\ell. \tag{61}$$

This means that $\mathcal{Y}^{j\pm\frac{1}{2}}_\ell$ are same eigenfunctions of the both operators $\tilde{\gamma}^r$ and $\hat{K}$. However we can use (59) to write

$$\tilde{\gamma}^r \psi = i \begin{pmatrix} O & \tilde{\sigma}^r \\ -\tilde{\sigma}^r & O \end{pmatrix} \begin{pmatrix} \xi_\pm(r)\mathcal{Y}^{j\pm\frac{1}{2}}_\ell \\ \zeta_\mp(r)\mathcal{Y}^{j\mp\frac{1}{2}}_\ell \end{pmatrix} e^{-i\omega t} = i \begin{pmatrix} -\zeta_\mp\mathcal{Y}^{j\pm\frac{1}{2}}_\ell \\ \xi_\pm\mathcal{Y}^{j\mp\frac{1}{2}}_\ell \end{pmatrix} e^{-i\omega t}. \tag{62}$$

Before determining explicit form of the radial parts of the Dirac spinors, we should discuss important property of eigenfunctions of anti-fermions which are generated from fermion eigenfunctions via charge conjugation operator.

3.1 Charge Conjugation

In the quantum field theories in curved space, concept of charge conjugation is accepted to be same as one which is in flat space quantum field theory. By regarding the convention (5), it is given by [43]

$$\psi^c = \tilde{\gamma}^y \psi^*, \tag{63}$$

where $\mathcal{C} = i\tilde{\gamma}^t\tilde{\gamma}^y$ is charge conjugation operator and ψ^c is Dirac wave function of anti-fermion for each fermion ψ with complex conjugate ψ^*. There is proven both of ψ^c and ψ satisfy the Dirac equation of motion in absence and also in presence of the electromagnetic fields but with opposite sign of charge (see for instance [43]).It is easy to check that $(\tilde{\gamma}^y)^* = \tilde{\gamma}^y$ for which (63) can be rewritten as $(\psi^c)^* = \tilde{\gamma}^y\psi$. This means that by having exact form of the Dirac waves for fermions we in fact can generate corresponding Dirac waves for ani-fermions via $(\psi^c)^* = \tilde{\gamma}^y\psi$. Now, we investigate radial parts of the Dirac waves, i.e., $u(r)$ and $v(r)$ given by (55).

3.2 Radial dependent part of the Dirac spinors

Substituting (55), into the Schrodinger like of Dirac equation (39) and by regarding the identities (49), (58), (59) and (62) we find two coupled ordinary differential equations for radial part of the Dirac field in which spherical Harmonic functions drop from two sides of the equations, such that

$$(\omega + qA_0(r) + m_\psi)\xi_\pm = \left[\frac{(1-e^{-V}+\kappa_\mp e^U)}{r} - \frac{U'}{2}e^{-V} - e^{U-V}\partial_r\right](i\zeta_\mp)$$
$$\left[\frac{(1-e^{-V}+\kappa_\pm e^U)}{r} - \frac{U'}{2}e^{-V} - e^{U-V}\partial_r\right]\xi_\pm = (\omega + qA_0(r) - m_\psi)(i\zeta_\pm). \tag{64}$$

Adding side by side of the above equations we obtain suitable forms of the equations such that

$$\left[\frac{1-e^{-V}-(1-\lambda)e^U}{r} - \frac{U'}{2}e^{-V} - e^{U-V}\frac{d}{dr} - (\omega + qA_0(r))\right]\chi_1^{\kappa_+} = m_\psi\chi_2^{\kappa_+}, \tag{65}$$

for $\kappa_+ = \lambda - 1$ and

$$\left[\frac{1-e^{-V}-(1+\lambda)e^U}{r} - \frac{U'}{2}e^{-V} - e^{U-V}\frac{d}{dr} - (\omega + qA_0(r))\right]\chi_1^{\kappa_-} = m_\psi\chi_2^{\kappa_-}, \tag{66}$$

for $\kappa_- = -(\lambda+1)$ respectively where we defined new spinors

$$\chi_1^{\kappa_\mp} = \xi_\mp + i\zeta_\pm, \quad \chi_2^{\kappa_\mp} = \xi_\mp - i\zeta_\pm. \tag{67}$$

These equations can be solved to present explicit form of the functions $\xi(r)$ and $\zeta(r)$ if the Maxwell field $A_0(r)$ and the metric fields $U(r)$ and $V(r)$ are known. Thus we should find explicit form of the Einstein equations and the Maxwell equations. Then, we solve them together. To have right side of the metric equation (19) we need explicit form of the stress tensors (21), (22) and (23). After some mathematical calculations they are found as follows respectively.

4 Stress tensors

In this section, we try to find stress tensor components of the Dirac field and the Maxwell field for spherically symmetric static Dirac star.

4.1 Dirac star stress tensor

Although straightforward way to find explicit form of the Dirac field stress tensor for the line element (29) is substitution of the obtained results above into the stress tensor (23) and these calculations are very long and tedious, fortunately there are details in scientific papers for instance [55]. At a first step, we rewrite $\phi_{1,2}$ given by the Dirac spinor (59) versus their two components with the following ansatz

$$\xi_{\pm}(r)\mathcal{Y}_{\ell}^{j\pm\frac{1}{2}} = \begin{pmatrix} -f(r)\mathcal{Y}_{\ell}^{j\pm\frac{1}{2}} \\ \mp if(r)\mathcal{Y}_{\ell}^{j\mp\frac{1}{2}} \end{pmatrix}, \quad \zeta_{\mp}(r)\mathcal{Y}_{\ell}^{j\mp\frac{1}{2}} = \begin{pmatrix} -ig(r)\mathcal{Y}_{\ell}^{j\pm\frac{1}{2}} \\ \mp g(r)\mathcal{Y}_{\ell}^{j\mp\frac{1}{2}} \end{pmatrix}, \tag{68}$$

for which we can find

$$\chi_1^{\mp}(r)\mathcal{Y}_{\ell}^{j\mp\frac{1}{2}} = \begin{pmatrix} (g-f)\mathcal{Y}_{\ell}^{j\mp\frac{1}{2}} \\ \pm i(f+g)\mathcal{Y}_{\ell}^{j\pm\frac{1}{2}} \end{pmatrix}, \quad \chi_2^{\mp}(r)\mathcal{Y}_{\ell}^{j\mp\frac{1}{2}} = \begin{pmatrix} -(g+f)\mathcal{Y}_{\ell}^{j\mp\frac{1}{2}} \\ \pm i(f-g)\mathcal{Y}_{\ell}^{j\pm\frac{1}{2}} \end{pmatrix}. \tag{69}$$

The particular choice (68) causes that the static solutions of the Dirac star is satisfied because the associated flux of particles, momentum density and density current of particles vanish and so the stress tensor (23) is time independent (see [55]). With this setting, non-vanishing of the Dirac field stress tensor components are

$$\begin{aligned}
&(T^{Dirac})_t^t = e^{-V}(fg' - gf') + \frac{2fg}{r} + m_\psi(f^2 - g^2) \equiv \omega e^{-U}(f^2 + g^2), \\
&(T^{Dirac})_r^r = e^{-V}(fg' - f'g), \\
&(T^{Dirac})_\theta^\theta = (T^{Dirac})_\varphi^\varphi = \frac{fg}{r},
\end{aligned} \tag{70}$$

where equivalency identity for time-time component of stress tensor is proven in the ref. [55]. Furthermore the four current density of Dirac particles is obtained versus the particle density as

$$j^\mu = q\rho_p\delta_\mu^0, \quad \rho_p = i\bar{\psi}\gamma^0\psi = 2(f^2 + g^2), \tag{71}$$

where the orthogonality condition on the spherical Harmonic eigenfunctions is applied. In the next subsection we calculate non-vanishing components of the Maxwell stress tensor and the Maxwell field equation.

4.2 Spherical Maxwell field stress tensor

We will now determine explicit form of the Maxwell field for the line element (29). It is easy to show that for $A_\mu = A_0(r)\delta_\mu^0$, the Maxwell equation (26) reads

$$\begin{aligned}
U''' = {} & \frac{V'}{r^2} - \frac{U'}{r^2} + \frac{V''}{r} - \frac{3U''}{r} - \frac{3V'^2}{r} + \frac{4U'V'}{r} - \frac{U'^2}{r} + 4U''V' + U'V'' \\
& - U'U'' - 3U'V'^2 + 2V'U'^2 + U'^3 + \frac{E'}{E}\left(\frac{V'}{r} - \frac{U'}{r} - U'' - U'^2 - U'V'\right) \\
& - \frac{e^{2V}}{4\alpha}\left(\frac{E'}{E} + \frac{2}{r} - V' - U' - \frac{4\beta q e^{2(U+V)}}{E}(f^2 + g^2)\right),
\end{aligned} \tag{72}$$

where with $E(r) = qA_0'(r)$, the four current density is substituted by (71). The Maxwell stress tensor (21) and (22) read

$$T^t_{t\,EM} = T^r_{r\,EM} = \frac{e^{-2(U+V)}E^2}{4},$$

$$T^{\theta}_{\theta\,EM} = T^{\varphi}_{\varphi\,EM} = -\frac{e^{-2(U+V)}E^2}{4}. \tag{73}$$

4.3 Interaction stress tensor

By some mathematical calculation and regarding the above results for the line element (29) we find components of the interaction stress tensor as follows.

$$\begin{aligned}
\Theta^t_t &= -E^2 e^{-2U-4V}\left[U'' + (U'-V')(U'+\frac{1}{r})\right],\\
\Theta^r_r &= E^2 e^{-2(U+V)}\Bigg\{\frac{U''}{2} + \frac{V''}{2} - \frac{E''}{E} - U'^2 - 2U'V' + \frac{3U'E'}{E} + \frac{2U'}{r} - V'^2,\\
&+ \frac{3V'E'}{E} + \frac{2V'}{r} - \frac{E'^2}{E^2} - \frac{2E'}{rE} + \frac{1}{r^2}\\
&\quad - e^{-2V}\left[U'' + U'^2 + \frac{3U'}{r} + \frac{V'}{r} - U'V'\right]\Bigg\},\\
\Theta^\theta_\theta &= \Theta^\phi_\phi = -E^2 e^{-2U-4V}\left(\frac{U'+V'}{r}\right).
\end{aligned} \tag{74}$$

4.4 Einstein tensor

To solve the field equations, we need the non-vanishing Einstein tensor components for the line element (29) which can be calculated straightforwardly as follows

$$\begin{aligned}
G^t_t &= \frac{1}{r^2} + \mathrm{e}^{-2V}\left(\frac{2V'}{r} - \frac{1}{r^2}\right),\\
G^r_r &= \frac{1}{r^2} - \mathrm{e}^{-2V}\left(\frac{2U'}{r} + \frac{1}{r^2}\right),\\
G^\theta_\theta &= G^\phi_\phi = \mathrm{e}^{-2V}\left(V'U' - U'^2 - U'' + \frac{V'}{r} - \frac{U'}{r}\right).
\end{aligned} \tag{75}$$

Composing the results above we now investigate the analytic solution of the fields.

5 Spherical setting of the field equations

By having the previous results for stress tensors and field equations described for the spherically symmetric static curved line element (29), we are ready now to solve them synchronously. The Dirac field equations given by (65) and (66) reduce to the following forms by substituting (69):

$$\begin{aligned}
\frac{f'}{f} &= \frac{e^{V-U} - e^{-U} + (\lambda-1)e^V}{r} - \frac{U'}{2}e^{-U} - (m_\psi + \omega + qA_0(r))e^{V-U},\\
\frac{g'}{g} &= \frac{e^{V-U} - e^{-U} + (\lambda-1)e^V}{r} - \frac{U'}{2}e^{-U} + (m_\psi - \omega - qA_0(r))e^{V-U},
\end{aligned} \tag{76}$$

for κ_+ and

$$\frac{f'}{f} = \frac{e^{V-U} - e^{-U} - (\lambda+1)e^V}{r} - \frac{U'}{2}e^{-U} - (m_\psi + \omega + qA_0(r))e^{V-U},$$

$$\frac{g'}{g} = \frac{e^{V-U} - e^{-U} - (\lambda+1)e^{V}}{r} - \frac{U'}{2}e^{-U} + (m_{\psi} - \omega - qA_0(r))e^{V-U}, \tag{77}$$

for κ_- respectively. The total stress tensor components in right hand side of the Einstein metric field equation (19) are obtained by substituting (70), (73) and (74), such that

$$\begin{aligned}\rho_E(r) = T^{t}_{t\ EM} + \alpha\Theta^{t}_{t} + \beta(T^{Dirac})^{t}_{t} = \beta\omega e^{-U}(f^2+g^2)\\ + \frac{E^2 e^{-2(U+V)}}{4}\left\{1 - 4\alpha e^{-2V}\left[U'' + (U'-V')\left(U' + \frac{1}{r}\right)\right]\right\},\end{aligned} \tag{78}$$

is the total energy density function and

$$\begin{aligned}p_r(r) = T^{r}_{r\ EM} + \alpha\Theta^{r}_{r} + \beta(T^{Dirac})^{r}_{r} = \beta e^{-V}(fg' - f'g) + \frac{E^2e^{-2(U+V)}}{4}\\ + \alpha E^2 e^{-2(U+V)}\left[\frac{U''}{2} + \frac{V''}{2} - \frac{E''}{E} - U'^2 - V'^2 - 2U'V' + \frac{3U'E'}{E}\right.\\ \left. + \frac{3V'E'}{E} - \frac{E'^2}{E^2} + \frac{2U'}{r} + \frac{2V'}{r} - \frac{2E'}{rE} + \frac{1}{r^2}\right]\\ - \alpha E^2 e^{-(2U+4V)}\left[U'' + U'^2 - U'V' + \frac{3U'}{r} + \frac{V'}{r}\right],\end{aligned} \tag{79}$$

is the radial pressure function and

$$\begin{aligned}p_t(r) &= T^{\theta}_{\theta\ EM} + \alpha\Theta^{\theta}_{\theta} + \beta(T^{Dirac})^{\theta}_{\theta}\\ &= \frac{\beta fg}{r} - \frac{e^{-2(U+V)}E^2}{4}\left[1 + 4\alpha e^{-2V}\left(\frac{U'+V'}{r}\right)\right],\end{aligned} \tag{80}$$

is the transverse pressure function. To have the Einstein metric equation components given by (19) we substitute (75), (78), (79) and (80) into it such that

$$U'' = (V'-U')\left(U' + \frac{1}{r}\right) + \frac{e^{2V}}{4\alpha} + \frac{e^{2(U+V)}}{8\pi\alpha E^2}\left(\frac{2V'}{r} - \frac{(1+e^{2V})}{r^2}\right) - \frac{\beta\omega e^{U+4V}}{8\pi\alpha E^2}(f^2+g^2), \tag{81}$$

$$\begin{aligned}(1-2e^{2V})U'' + V'' - \frac{2E''}{E} = -\frac{1}{2\alpha} - \frac{2}{r^2} + 2U'^2 + 4U'V' - \frac{4U'E'}{E} - \frac{4U'}{r}\\ + 2V'^2 - \frac{6V'E'}{E} - \frac{4V'}{r} + \frac{2E'^2}{E^2} + \frac{4E'}{rE}\\ + 2e^{2V}\left(U'^2 - U'V' + \frac{3U'}{r} + \frac{V'}{r}\right) - \frac{e^{2U+4V}}{8\pi\alpha r^2E^2}\\ + \frac{e^{2(U+V)}}{8\pi\alpha E^2}\left(\frac{2U'}{r} + \frac{1}{r^2}\right) + \frac{\beta e^{2U+3V}}{\alpha E^2}(fg' - f'g),\end{aligned} \tag{82}$$

and

$$U'' = U'V' - U'^2 + \frac{V'}{r} - \frac{U'}{r} + \frac{8\pi\beta fge^{2V}}{r} - 2\pi E^2e^{-2U}\left[1 + 4\alpha e^{-2V}\left(\frac{U'+V'}{r}\right)\right]. \tag{83}$$

We are ready to solve the Maxwell equation (72), the Dirac equations (76), (77), and three components of the Einstein field equations (81), (82) and (83). They are six equations which should give us five un-known fields U, V, E, f, g. In fact one of these equations are a constraint condition between the solutions. This clime is because of the energy momentum conservation $\nabla_\mu T^\mu_\nu = 0$ or equivalently the Bianchi identity $\nabla_\mu G^\mu_\nu = 0$ which relates three components of the Einstein tensors to each other. Usually the stress energy conservation of a stellar fluid is called as Tolman-Oppenheimer-Volkoff (TOV) which for the line element (29) leads

$$p_r' - U'\rho_E + \left(U' + \frac{2}{r}\right)p_r - \frac{2p_t}{r} = 0, \tag{84}$$

in which total energy density ρ_E and directional pressures $p_{r,t}$ should be substituted by (78) and (79) and (80). This equation can be solved if there are relations between the energy density and the directional pressures. For all of relativistic isotropic stars in which $p_r = p_t = p$, the equation of state is polytropic such that $p(\rho) = K\rho^\gamma$ for which the constant K is called barotropic index and $\gamma = \frac{C_P}{C_V}$. In this, C_P and C_V are heat capacity at constant pressure and at constant volume respectively. However we do not investigate to solve the TOV equation above as in our previous work [57], but we take advantage of the fact that the components of the momentum energy are related to each other in a similar way to the components of the Einstein tensor via the Bianchi identity, and hence one of the equations of motion is a constraint, and we will leave aside one of them in the following.
It is obvious that the field equations above are nonlinear and so we have two different methods ahead to find their solutions, i.e., (a) numerical method and (b) perturbation method. We choose (b) method in what follows. To do this, we separate two different classes of solutions corresponding with $\kappa_+ = \lambda - 1$ and $\kappa_- = -(\lambda + 1)$, but each of the latter cases have two different solutions of the field if the fermions and anti-fermions have spin up or spin down. Thus at all, the physical state of such a Dirac spherically symmetric star will determined with four different classes of solutions. In this paper, we investigate one of these situations and other cases are dedicated for our future works. The case, which we like to solve the field equations at below, is for fermions/antifermions stellar matter which have spin up only. This restrict us to choose $f \neq 0$ with a ansatz trivial solution $g = 0$ for the Dirac equation (76).

5.1 Solutions of the fields for $\kappa_+ = \lambda - 1, f \neq 0, g = 0$

In this case to find U, V, f, E we need 4 differential equations which we choose (72), (76), (81), and (82). It is useful we use other equations instead of (81), and (82) by substituting U'' given by and (83). In that case, (82) reads

$$\begin{aligned} V'' - \frac{2E''}{E} &= -\frac{1}{2\alpha} - \frac{2}{r^2} + 3U'V' + 3U'^2, +2V'^2 - \frac{5V'}{r} - \frac{3U'}{r} + 4e^{2V}\left(\frac{U' + V'}{r}\right) \\ &+ 2\pi E^2 e^{-2U}(1 - e^{2V})\left[1 + 4\alpha e^{-2U}\left(\frac{U' + V'}{r}\right)\right] + \frac{E'}{E}\left(\frac{4}{r} + \frac{2E'}{E}\right. \\ &\left. - 6V' - 4U'\right) + \frac{e^{2(U+V)}}{8\pi\alpha E^2}\left[\frac{2U'}{r} + \frac{1 - e^{2V}}{r^2}\right], \end{aligned} \tag{85}$$

and (81) reduces to the following constraint condition.

$$\begin{aligned} &\left[8\pi\alpha E^2 e^{-2V} + \frac{e^{2U}}{4\pi\alpha E^2}\right]\frac{V'}{r} + 8\pi\alpha E^2 e^{-2V}\frac{U'}{r} \\ &+ \frac{1}{4\alpha} + 2\pi E^2 - \frac{e^{2U}}{8\pi\alpha E^2}\left[\frac{1 + e^{2V}}{r^2} + \beta\omega f^2 e^{2V}\right] = 0. \end{aligned} \tag{86}$$

Substituting V'' and U'' given by (85) and (83), the Maxwell equations (72) reads

$$U''' = \frac{V'}{r^2} - \frac{U'}{r^2} + U'^3 + 2V'U'^2 - 3U'V'^2 - \frac{U'^2}{r} + \frac{4U'V'}{r} - \frac{3V'^2}{r}$$
$$+ \left(\frac{1}{r} + U'\right)\Bigg\{ - \frac{1}{2\alpha} + \frac{4U'e^{-2V}}{r} + 2\pi E^2 e^{-2U}(e^{-2V} - 1)\Bigg[1$$
$$+ 4\alpha e^{-2V}(\frac{V' + U'}{r})\Bigg] - 2(V' + U')\left[V' + U' + \frac{E'}{E}\right]$$
$$+ \frac{e^{2(U+V)}}{4\pi\alpha E^2}\left[\frac{1}{r^2} - e^{2V}\left(\frac{2U'}{r} + \frac{1}{r^2}\right)\right]\Bigg\} + \frac{E'}{E}\left(\frac{V'}{r} - \frac{U'}{r} - U'^2 - U'V'\right)$$
$$+ \frac{e^{2V}}{4\alpha}\left[\frac{E'}{E} + \frac{2}{r} - V' - U' + \frac{4\beta q f^2 e^{2(U+V)}}{E^2}\right]. \tag{87}$$

Thus we will use the equations (85), (86), (87) together with the Dirac equation (76) to study formation of a Dirac star in what follows. To do this, we use the dynamical system approach which is mentioned in the introduction section.
At a first step, we collect set of dimensionless first order differential equations generated from the field equations (76),(85), (86) and (87). Hence, to have some simpler forms for the field equations, we define a dimensionless logarithmic radial variable τ and some new dimensionless fields as follows.

$$r = \sqrt{\alpha}e^{\tau}, \quad \frac{d}{d\tau} = \dot{}, \quad f = \frac{e^{\delta}}{\sqrt{32\pi\beta q\alpha\sqrt{\alpha}}}, \quad \bar{\omega} = \frac{\omega}{32\pi q\sqrt{\alpha}},$$
$$\bar{m} = \sqrt{\alpha}(m_{\psi} + \omega), \quad \Pi(\tau) = \sqrt{\alpha}qA_0, \quad \sqrt{8\pi\alpha}E = e^{\epsilon}, \quad \sigma = \frac{q}{\sqrt{8\pi\alpha}}, \tag{88}$$

in that case the equation (76) reduces to the following form

$$\dot{\delta} + \frac{\dot{U}}{2}e^{-U} = e^{V-U} - e^{-U} + (\lambda - 1)e^{V} - (\bar{m} + \Pi)e^{\tau+V-U}, \tag{89}$$

with

$$\dot{\Pi} = \sigma e^{\epsilon}. \tag{90}$$

The equation (85) reads

$$\ddot{V} - 2\ddot{\epsilon} = 2\dot{\epsilon} + 4\epsilon^2 - 2 - \frac{e^{2\tau}}{2} + 3\dot{U}\dot{V} + 3\dot{U}^2 + 2\dot{V}^2 - 4\dot{V} - 3\dot{U} + 4(\dot{U} + \dot{V})e^{2V}$$
$$+ \frac{e^{2(\epsilon+\tau-U)}}{4}(1 - e^{2V})[1 + 4e^{-2(U+\tau)}(\dot{U} + \dot{V})] - 6\dot{\epsilon}\dot{V} - 4\dot{\epsilon}\dot{U}$$
$$+ [1 + 2\dot{U} - e^{2V}]e^{2(U+V-\epsilon)}. \tag{91}$$

The equation (86) is

$$\dot{U} + [1 + 2e^{2(U+V-2\epsilon)}]\dot{V} = \bar{\omega}e^{2(U-2\epsilon+2V+\delta+\tau)} + e^{2(U-2\epsilon+2V)}$$
$$+ e^{2(U+V-2\epsilon)} - \frac{e^{2(\tau+V-2\epsilon)}}{4} - \frac{e^{2(\tau-\epsilon+V)}}{4}, \tag{92}$$

and finally for the Maxwell equation (87) we find

$$\dddot{U} - 3\ddot{U} = -3\dot{U} + \dot{V} + \dot{U}^3 + 2\dot{V}\dot{U}^2 - 3\dot{U}\dot{V}^2 - \dot{U}^2 + 4\dot{U}\dot{V} - 3\dot{V}^3$$

$$
\begin{aligned}
&- e^{2\tau}(1+\dot{U})(e^{-2V}-1)[1+4e^{-2(\tau+V)}(\dot{V}+\dot{U})] \\
&- 2(1+\dot{U})(\dot{U}+\dot{V})[\dot{U}+\dot{V}+\dot{\epsilon}] \\
&+ 2(1+\dot{U}e^{2(U+V-\epsilon)})[1-e^{2V}(1+2\dot{U})]+\dot{\epsilon}(\dot{V}-\dot{U}-\dot{U}^2-\dot{U}\dot{V}) \\
&+ \frac{e^{2(\tau+U)}}{4}[\dot{\epsilon}+2-\dot{V}-\dot{U}+e^{2(\tau+U+V+\delta-\epsilon)}]. \qquad (93)
\end{aligned}
$$

Before to generate eight dimensional phase space of the system under consideration and whose first order differential equations, we make two very important conditions to simplify the above equations and then proceed to form eight differential equations in phase space as follows:

- It is obvious that the above equations are non-autonomous and coefficients of some of the equations above diverges to infinity because at $\tau \to \infty$ corresponds to $r >>> \sqrt{\alpha}$. Thus we consider solutions of the above equations just at small scales $r << \sqrt{\alpha}$ for which $\tau \to -\infty$. In the latter case all coefficients of the equations above containing the exponential factor e^{τ} vanish and so the equations are autonomous.
- In fact, linear order of the fields equation are applicable in the Jacobian of the Eq. (1) in the dynamical system approach and higher order perturbation terms are negligible.

Regarding the above two important rules have some useful beneficial consequences. The length of the equations suppress without to omitted physical contents of the problem. If we have possibly stable Dirac stars then, they have a finite size and so we must solve the above complicated nonlinear equations in the limits $\tau \to -\infty$ or equivalently at small scales $r << \sqrt{\alpha}$. Thus we continue our investigations by regarding the above two rules by substituting the perturbation series functions

$$e^U \sim 1+U+\cdots, \quad e^V \sim 1+V+\cdots, \quad e^\epsilon \sim 1+\epsilon+\cdots \qquad (94)$$

into the above equations. In that case, the equation (89) reduces to the following form.

$$\dot{\delta}+\frac{\dot{U}}{2} \approx \lambda-1+\lambda V. \qquad (95)$$

The equation (90) reaches to the following form.

$$\dot{\Pi} \approx \sigma+\sigma\epsilon. \qquad (96)$$

The equation (91) is

$$
\begin{aligned}
\ddot{V}-2\ddot{\epsilon} \approx& -2+2\dot{\epsilon}+4\epsilon^2+3\dot{U}\dot{V}+3\dot{U}^2+2\dot{V}^2+3\dot{U}+4\dot{V}+12\dot{U}V \\
&+8\dot{V}V-6\dot{\epsilon}\dot{V}-4\dot{\epsilon}\dot{U}-2V+4\dot{U}U-4UV-4V^2+4V\epsilon. \qquad (97)
\end{aligned}
$$

The equation (92) reads

$$\dot{U}+\dot{V} \approx 2+4U+6V-8\epsilon, \qquad (98)$$

and finally the equation (93) takes the following form.

$$
\begin{aligned}
\dddot{U}-3\ddot{U} \approx& -7\dot{U}-3\dot{V}-\dot{U}^3-4\dot{U}^2\dot{V}-5\dot{U}\dot{V}^2-6\dot{U}^2-5\dot{V}^2-2\dot{U}\dot{\epsilon}-2\dot{V}\dot{\epsilon} \\
&-2\dot{U}^2\dot{\epsilon}-2\dot{U}^2\dot{V}-2\dot{U}\dot{V}\dot{\epsilon}-20V\dot{U}-4V\dot{U}^2-8V\dot{U}+8\epsilon\dot{U}
\end{aligned}
$$

$$-8UV\dot{U}-8V^2\dot{U}+8V\epsilon\dot{U}-16V^2\dot{U}^2-16UV\dot{U}^2+16V\epsilon\dot{U}^2. \tag{99}$$

By according to the dynamical system approach given by the eq. (1) we make eight dimensional phase space describing the vector field

$$\vec{\Lambda}=\{\Pi,\epsilon,V\delta,U,V,X,Y,Z\}, \tag{100}$$

in which X,Y,Z are assumed to be

$$X=3\dot{U}-3U+V,\quad Y=\dot{V}-2\dot{\epsilon},\quad \dot{X}=3Z, \tag{101}$$

and find first order differential equations for each of these fields via (95), (96), (97), (98), (99) and (101). By using (95), (98), and (101) we find

$$\begin{aligned}
\dot{\epsilon}&=-\frac{X}{6}+\frac{19V}{6}+\frac{3U}{2}-\frac{Y}{2}-4\epsilon+1,\\
\dot{\delta}&=\lambda V-\frac{U}{2}+\frac{V}{6}-\frac{X}{6}+\lambda-1,\\
\dot{U}&=U-\frac{V}{3}+\frac{X}{3},\\
\dot{V}&=-\frac{X}{3}+\frac{19V}{3}+3U-8\epsilon+2.
\end{aligned} \tag{102}$$

First order differential equations for Y and Z are obtained via (102) and (101), such that

$$\begin{aligned}
&\dot{Y}=G_7(\epsilon,U,V,X,Y)\\
&G_7=4-60\epsilon^2+4\left(-2+10U+11V-2X-6Y\right)\epsilon\\
&+U^2+\left(8-4V+6X+11Y\right)U+\frac{7V^2}{9}\\
&+\frac{\left(64X+165Y+168\right)V}{9}+\frac{X^2}{9}+\frac{\left(4-Y\right)X}{3}+5Y
\end{aligned} \tag{103}$$

$$\begin{aligned}
&\dot{Z}=G_8(\epsilon,U,V,X,Y)\\
&G_8=-30+216\epsilon+2Y-2X-160V-114U-\frac{32}{3}V^2\epsilon U+\frac{1832}{9}VX\epsilon\\
&-\frac{16}{9}UVX^2+\frac{4}{3}UXY-\frac{32}{3}U^2VX-\frac{32}{9}V^2\epsilon X+\frac{304}{3}UX\epsilon\\
&+\frac{16}{9}V\epsilon X^2-\frac{772}{9}XUV+\frac{1448}{3}UV\epsilon+16V\epsilon U^2-\frac{8}{3}X\epsilon Y+\frac{8}{3}V\epsilon Y\\
&-8U\epsilon Y+2VXY-\frac{64}{9}UV^2X+\frac{14}{3}UVY+\frac{16}{9}V^3\epsilon+344U^2\epsilon\\
&-2V^2Y+4U^2Y-128X\epsilon^2+128V\epsilon^2-384U\epsilon^2+\frac{26}{9}VX^2+\frac{20}{9}UX^2\\
&-\frac{724}{9}XV^2-\frac{56}{3}XU^2-\frac{1264}{9}UV^2-\frac{754}{3}VU^2+\frac{80}{9}UV^3\\
&-\frac{16}{3}U^2V^2-16U^3V-\frac{16}{9}V^2X^2+\frac{32}{9}V^3X-\frac{40}{9}X^2\epsilon-\frac{1792}{9}V^2\epsilon+\frac{2}{3}XY\\
&+\frac{16}{3}VY+6UY-8\epsilon Y-18UX-149U^2-\frac{316}{9}VX-\frac{1}{9}X^2+\frac{112}{3}X\epsilon\\
&+496U\epsilon-\frac{1627}{9}V^2-384\epsilon^2+\frac{1616}{3}V\epsilon-376UV
\end{aligned}$$

$$+\frac{32}{3}V\epsilon UX+\frac{698}{9}V^3-76U^3-\frac{16}{9}V^4. \tag{104}$$

We now can use the above mathematical calculations to show the vector field in phase space (100) with the differential equation (1), such that

$$\frac{d}{d\tau}\begin{pmatrix}\Pi\\ \epsilon\\ \delta\\ U\\ V\\ X\\ Y\\ Z\end{pmatrix}=\begin{pmatrix}\sigma+\sigma\epsilon\\ -\frac{X}{6}+\frac{19V}{6}+\frac{3U}{2}-\frac{Y}{2}-4\epsilon+1\\ \lambda V-\frac{U}{2}+\frac{V}{6}-\frac{X}{6}+\lambda-1\\ U-\frac{V}{3}+\frac{X}{3}\\ -\frac{X}{3}+\frac{19V}{3}+3U-8\epsilon+2\\ 3Z\\ G_7(\epsilon,U,V,X,Y)\\ G_8(\epsilon,U,V,X,Y)\end{pmatrix}. \tag{105}$$

There are two class of critical points in phase space which are obtained by solving the equations

$$\dot{\Pi}=0=\dot{\epsilon}=\dot{\delta}=\dot{U}=\dot{V}=\dot{X}=\dot{Y}=\dot{Z}, \tag{106}$$

such that

$$P_c^{\pm}:\quad \Lambda_c^i=\begin{pmatrix}\Pi_c(\sigma=0)=0\\ \epsilon_c=(-\lambda+1\pm\sqrt{\lambda^2+2\lambda-1})/2\lambda\\ \delta_c=arbitrary\\ U_c=(-1\pm2\sqrt{\lambda^2+2\lambda-1})/2\lambda\\ V_c=-(\lambda-1)/\lambda\\ X_c=-(2\lambda-5\pm6\sqrt{\lambda^2+2\lambda-1})/2\lambda\\ Y_c=0\\ Z_c=0\end{pmatrix}, \tag{107}$$

in which

$$\lambda=j+\frac{1}{2}=1,2,3,4\cdots. \tag{108}$$

Calculating the Jacobi matrix components at the critical points above is straightforward via $J_{ij}=\frac{\partial\dot{\Lambda}_i}{\partial\Lambda_j}$ such that

$$J_{ij}^{\pm}=\begin{pmatrix}0&0&0&0&0&0&0&0\\ -4&0&\frac{3}{2}&\frac{19}{6}&-\frac{1}{6}&-\frac{1}{2}&0&0\\ 0&0&-\frac{1}{2}&\lambda+\frac{1}{6}&-\frac{1}{6}&0&0&0\\ 0&0&1&-\frac{1}{3}&\frac{1}{3}&0&0&0\\ -8&0&3&\frac{19}{3}&-\frac{1}{3}&0&0&0\\ 0&0&0&0&0&0&3&0\\ J_{71}&0&J_{73}&J_{74}&J_{75}&-1&0&0\\ 24&0&J_{83}&J_{84}&J_{85}&0&0&0\end{pmatrix}, \tag{109}$$

in which

$$J_{71}^{\pm}=\frac{\pm4\sqrt{\lambda^2+2\lambda-1}+16\lambda-56}{\lambda},$$
$$J_{73}^{\pm}=\frac{\pm4\sqrt{\lambda^2+2\lambda-1}-14\lambda+30}{\lambda},$$

$$
\begin{aligned}
J_{74}^{\pm} &= \frac{2(65-18\lambda \mp 5\sqrt{\lambda^2+2\lambda-1})}{3\lambda},\\
J_{75}^{\pm} &= \frac{\pm 4\sqrt{\lambda^2+2\lambda-1}-6\lambda+2}{3\lambda},\\
J_{83}^{\pm} &= \frac{\pm 4\left(2\lambda-1\right)\sqrt{\lambda^2+2\lambda-1}+4\lambda^2-20\lambda}{\lambda^2},\\
J_{84}^{\pm} &= \frac{\pm 4\left(-2\lambda+1\right)\sqrt{\lambda^2+2\lambda-1}-70\lambda^2+20\lambda}{3\lambda^2},\\
J_{85}^{\pm} &= \frac{\pm 4\left(-2\lambda+1\right)\sqrt{\lambda^2+2\lambda-1}-70\lambda^2+20\lambda}{3\lambda^2}.
\end{aligned} \tag{110}
$$

We now can find eigenvalues of the Jacobi matrix above by solving the secular equation $det(J_{ij}-s\delta_{ij})=0$, such that

$$
s^2\left(\frac{7}{2}s^3+\frac{5}{3}s^4-4s^2+\frac{7}{6}s^5+s^6-\frac{4}{3}\lambda\, s^3-4\lambda s-\lambda\, s^4-3\lambda\, s^2\right)=0, \tag{111}
$$

for both of $J_{ij}^{\pm}$ with the following parametric solutions.

s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
0	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	$A(\lambda)$	$B(\lambda)+iC(\lambda)$	$B(\lambda)-iC(\lambda)$

(112)

where we defined

$$
\begin{aligned}
A &= \frac{193+108\lambda-7g^{\frac{1}{3}}+g^{\frac{2}{3}}}{18g^{\frac{1}{3}}},\\
B &= -\frac{193+108\lambda+14g^{\frac{2}{3}}}{36g^{\frac{1}{3}}},\\
C &= -\frac{\sqrt{3}(193+108\lambda-g^{\frac{1}{3}})}{36g^{\frac{1}{3}}},\\
g &= 2754\lambda-1855+18\sqrt{-3888\lambda^3+2565\lambda^2-68784\lambda-11568}.
\end{aligned} \tag{113}
$$

Numeric values of the above parametric eigenvalues are determined by substituting the quantized numbers $\lambda = 1,2,3,4,\cdots$. When the eigenvalues are absolutely real negative (positive) numbers then the system is stable(unstable) but when they are complex numbers with negative (positive) real part then the system will be spiral stable(unstable) nature in phase space. Therefore appropriate to identify the stable regimes of the λ parameter by solving the secular equation (111) versus s parameter, such that

$$
\lambda = \frac{\left(6s^2+7s-8\right)s}{6s+8}, \tag{114}
$$

which its diagram in the figure (1(a)) shows stable regimes ($s<0$). For the Jacobi matrix (110) the equation (1) reads to the following set of differential equations.

$$
\begin{aligned}
\dot{\Pi} &= 0,\\
\dot{\epsilon} &= \frac{3}{2}\delta+\frac{19}{6}U-\frac{1}{6}V-\frac{1}{2}X,\\
\dot{\delta} &= -\frac{\delta}{2}+\left(\frac{1}{6}+\lambda\right)U-\frac{V}{6},
\end{aligned}
$$

$$\begin{aligned}
\dot{U} &= \delta - \frac{U}{3} + \frac{V}{3}, \\
\dot{V} &= 3\delta + \frac{19}{3}U - \frac{1}{3}V, \\
\dot{X} &= 3Y, \\
\dot{Y} &= J_{83}\delta + J_{74}U + J_{75}V - X, \\
\dot{Z} &= J_{83}\delta + J_{84}U + J_{85}V,
\end{aligned} \tag{115}$$

where the first equation shows that Π should be constant field which we used the critical value $\Pi_c = 0$. We show stability nature of the system with arrow diagrams of the fields equations in phase space in 1(c) and 1(d). We know from linear algebra theory that solutions of the equation (1) near the critical points are obtained as

$$\Lambda_i = \sum_{j=1}^{8} J_{ij}^{\pm}\Lambda_c^j e^{s_j\tau}, \tag{116}$$

in which Λ_c^j are components of the critical point (107) and s_j are eight eigenvalues are given by (112) and the Jacobi matrix J_{ij} should be substituted by (109). One can follow this to find explicit form of the vector field solution (116) as follows:

$$\begin{aligned}
&\Pi(\tau) = 0, \\
&\epsilon(\tau) \approx \frac{3}{2}\delta_c + \frac{(19U_c - V_c)}{6}\cos\sqrt{3}\tau - \frac{X_c e^{(ReA)\tau}}{2}\cos(ImA\tau), \\
&\delta = constant, \\
&U(\tau) \approx \delta_c + \frac{(V_c - U_c)}{3}\cos\sqrt{3}\tau, \\
&V(\tau) \approx 3\delta_c + \frac{(19U_c - V_c)}{3}\cos\sqrt{3}\tau, \\
&X(\tau) = 0, \\
&Y(\tau) \approx J_{73}\delta_c + (J_{74}U_c + J_{75}V_c)\cos\sqrt{3}\tau - X_c e^{ReA\tau}\cos(ImA\tau), \\
&Z(\tau) \approx J_{83}\delta_c + (J_{84}U_c + J_{85}V_c)\cos\sqrt{3}\tau,
\end{aligned} \tag{117}$$

where we suppress higher order multiplications of the solutions which make second or higher order harmonics of the cosine functions and also we drop imaginary parts of the solutions above because their amplitude are negligible with respect to real parts. other reason which make to drop the imaginary parts is because that all physical fields for instance energy density and pressures should be real. In other words, the above solutions are leading order perturbations solutions around the critical points. Also in the above solutions ReA and ImA are real and imaginary parts of the parameter of A given by (113).
It is suitable to find explicit form of the energy density and directional pressures fluctuations near the critical points solutions above. They are found by substituting the obtained solutions (117) straightforwardly such that

$$\begin{aligned}
&\delta\bar{\rho}_E = 8\pi\alpha(\rho_E - \rho_{cE}) \approx \frac{\sqrt{3}}{4}\delta_c^2(2V_c - 70U_c)e^{-2\tau}\sin\sqrt{3}\tau, \\
&\delta\bar{p}_r = 8\pi\alpha(p_r - p_{rc}) \approx \frac{(V_c - U_c)}{2}e^{-2\tau}[\sqrt{3}\sin\sqrt{3}\tau + \cos\sqrt{3}\tau] \\
&+ \frac{X_c}{2}e^{(ReA-2)\tau}[1 + (ReA)^2 + (ImA)^2]\cos(ImA\tau)
\end{aligned}$$

$$+\frac{X_c(ImA)}{2}e^{(ReA-2)\tau}\sin(ImA\tau),$$
$$\delta\bar{p}_t = 8\pi\alpha(p_t - p_{tc}) \approx 6\sqrt{3}U_c e^{-2\tau}\sin\sqrt{3}\tau, \tag{118}$$

in which we defined critical energy density and directional critical pressures as follows.

$$8\pi\alpha\rho_{Ec} \approx \frac{1}{4} + 8\pi\bar{\omega}(1+2\delta_c) + \frac{\lambda + 16\pi\bar{\omega}(1\mp\sqrt{\lambda^2+2\lambda-1})}{4\lambda},$$
$$8\pi\alpha p_{rc} \approx \frac{23\lambda \mp 5\sqrt{\lambda+2\lambda-1}}{4\lambda},$$
$$8\pi\alpha p_{tc} = \frac{2-4\lambda \pm 3\sqrt{\lambda^2+2\lambda-1}}{4}. \tag{119}$$

We notify that for $\lambda = 1, 2, 3, \cdots$ the parameter A is a complex number. For instance for a Dirac star in the ground state $\lambda = 1$ corresponding with $j = \frac{1}{2}, S = \frac{1}{2}, \ell = 0$ we have

$$A_{\lambda=1} = 1.3 + 8.8 \times 10^{-6}i, \tag{120}$$

for which the numeric values of the critical points components are

$$P_c^{\pm}(\lambda=1): \quad \Lambda_c^i(\lambda=1) = \begin{pmatrix} \Pi_c = 0 \\ \epsilon_c^{\pm} = \pm\frac{\sqrt{2}}{2} \\ \delta_c \equiv constant \\ U_c = \frac{-1\pm\sqrt{2}}{2} \\ V_c = 0 \\ X_c = \frac{3\mp6\sqrt{2}}{2} \\ Y_c = 0 \\ Z_c = 0 \end{pmatrix}. \tag{121}$$

Substituting these numeric values into the solutions above we obtain exact form of the decreasing functions for energy density and the directional pressures, such that

$$\delta\bar{\rho}_E^{\pm} \approx \frac{70\sqrt{3}(1\mp\sqrt{2})\delta_c^2}{8}e^{-2\tau}\sin\sqrt{3}\tau,$$
$$\delta\bar{p}_r^{\pm} \approx \frac{1\mp\sqrt{2}}{4}e^{-2\tau}[\sqrt{3}\sin\sqrt{3}\tau + \cos\sqrt{3}\tau]$$
$$+\frac{8.1(1\mp 2\sqrt{2})}{4}e^{-0.7\tau}\cos(8.8\times10^{-6}\tau),$$
$$\delta\bar{p}_t^{\pm} \approx 3\sqrt{3}(-1\pm\sqrt{2})e^{-2\tau}\sin\sqrt{3}\tau, \tag{122}$$

with critical values

$$\bar{\rho}_{Ec}^{\pm} = 8\pi\alpha\rho_{Ec}^{\pm} \approx \frac{1}{2} + 4\pi\bar{\omega}(3+4\delta_c\mp\sqrt{2}),$$
$$\bar{p}_{rc}^{\pm} = 8\pi\alpha p_{rc}^{\pm} \approx \frac{23\mp5\sqrt{2}}{4},$$
$$\bar{p}_{tc}^{\pm} = 8\pi\alpha p_{tc}^{\pm} \approx \frac{-2\pm3\sqrt{2}}{4}. \tag{123}$$

We plot diagrams of the density and the pressures (123) in Figure 1(b) and Figures 2(a) and 2(b) respectively. Diagrams of the pressures and density show that the Dirac star under consideration does not have a surface with determined single radius same as Newtonian stars but the pressures suppress at infinity $r \to \infty$. This motivates that we evaluate an average radius of such a cloud matter as a compact star. This is done at below determine the scale Dirac star.

5.2 Size of the Dirac stars

One may ask a question about the size of such a Dirac relativistic star. Usually, Newton stars or neutron stars have sharp surface with finite radius which separates inside and outside of the star region. It is determined by setting $\rho(\mathcal{R}) = 0 = p_r(\mathcal{R})$ but for boson and fermion stars which have not sharp radius $\mathcal{R}$ their density and radial pressure suppress asymptotically to vanish at infinite distance [56] same as figures 1(b) and 2(a) and 2(b) in this work. Such matter distributions have estimated radius which can be calculated from the following equations

$$<\mathcal{R}>= -\frac{1}{Q}\int r dr^3\sqrt{g}j^t, \tag{124}$$

where

$$Q = -\int dr^3\sqrt{g}j^t, \tag{125}$$

is the Noether charge and j^t is time component of conserved density current (6). In fact, it originates from internal $U(1)$ gauge symmetry as $\psi \to e^{i\chi}\psi$ with constant gauge field χ. In other words, in the model with un-gauged $U(1)$ symmetry, the Noether charge Q is interpreted as the number of fermion-boson particles with mass m_ψ that make up the fermion-boson star. To calculate estimated radius of the obtained relativistic Dirac star (124), we substitute the obtained solution above into the relation (124) such that

$$\begin{aligned} \frac{<\mathcal{R}>}{\sqrt{\alpha}} &= \frac{\int_{-\infty}^{\infty} d\tau e^{4\tau+U+V+2\delta}}{\int_{-\infty}^{\infty} d\tau e^{3\tau+U+V+2\delta}} \\ &= \frac{\int_{-\infty}^{\infty} d\tau e^{4\tau+8U_c\cos\sqrt{3}\tau}}{\int_{-\infty}^{\infty} d\tau e^{3\tau+8U_c\cos\sqrt{3}\tau}} \\ &= \frac{\int_0^{\infty} d\tau \cosh(4\tau) e^{8U_c\cos\sqrt{3}\tau}}{\int_0^{\infty} d\tau \cosh(3\tau) e^{8U_c\cos\sqrt{3}\tau}} \end{aligned} \tag{126}$$

in which $U(\tau), V(\tau), \delta(\tau)$ are substituted by (117). The above integral equations have not analytic solutions and we must solve numerically by considering with ultraviolet/infrared cutoff length scales instead of the lower bound zero and upper bound infinity of the integral equations. But we consider two approximations to evaluate the above integral equations in what follows. I) we replace average of the function $e^{8U_c\cos\sqrt{3}\tau}$ because that remains finite at $0 < \tau < \infty$ such that

$$\overline{e^{8U_c\cos\sqrt{3}\tau}} = \frac{e^{8U_c} + e^{-8U_c}}{2} = \cosh(8U_c), \tag{127}$$

in which we used maximum and minimum values of the cosine function for $0 < \tau < \infty$. In this limit, $\cosh(8U_c)$ drops from numerator and denominator of the fraction and so (126) reduces to the following form

$$\frac{<\mathcal{R}>}{\sqrt{\alpha}} \approx \frac{\int_0^{\infty} d\tau \cosh(4\tau)}{\int_0^{\infty} d\tau \cosh(3\tau)}. \tag{128}$$

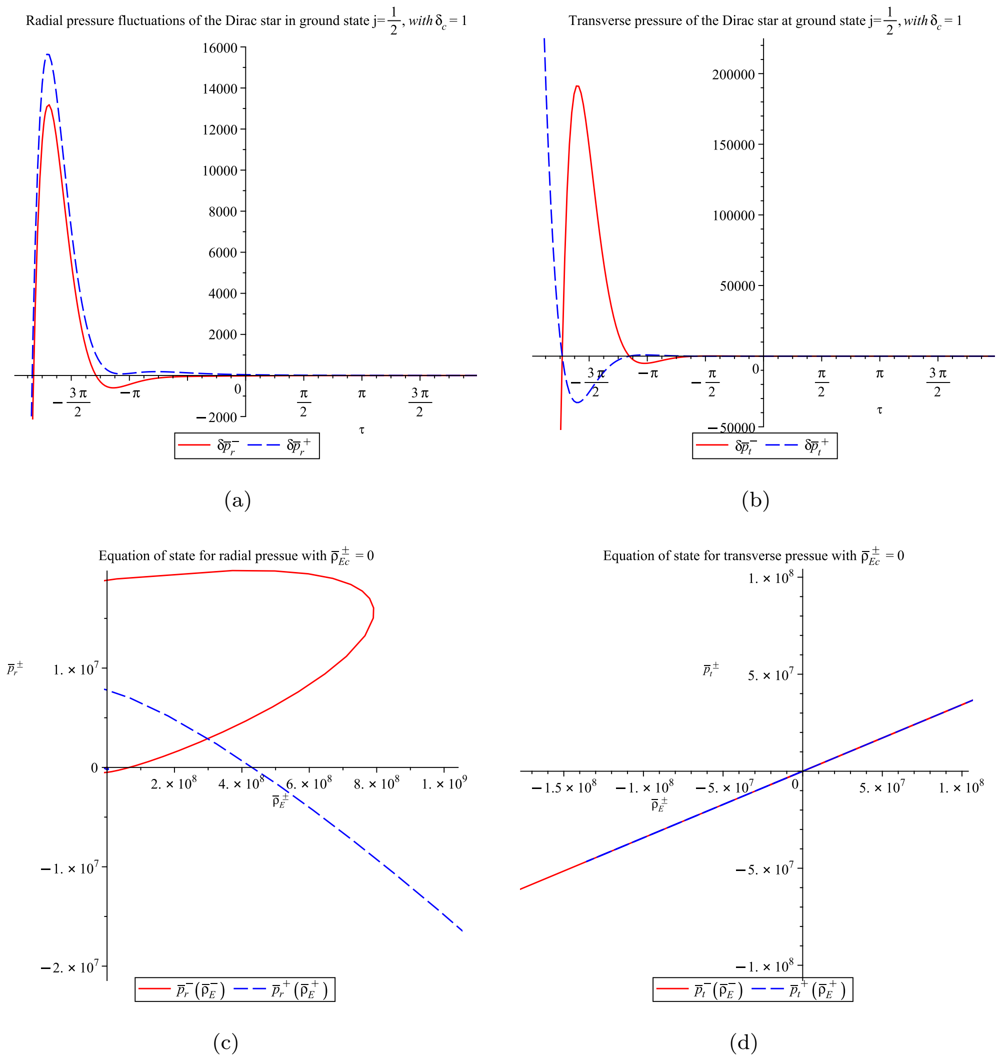

Figure 2: (a) Variation of radial pressure difference vs the radial coordinate. This shows that central pressure diverges to negative infinity but at large distances it vanishes for both of branches of the solutions (b) Variation of transverse pressure difference vs the radial coordinate, where central value diverge to positive infinity and suppresses to zero value at large distance. (c) Equation of state for radial pressure shows both of branches of the solutions have both dark invisible and visible behavior for the star. (d) Equation of state for transverse pressure behaves as barotropic behavior in both branches of the solutions.

II) in the second step we use transformations $4\tau = \zeta$ and $3\tau = \zeta$ in the numerator and in the denominator respectively, for which limit values of the integrals given by (128) drops again from numerator and denominator such that, one can infer that

$$\frac{<\mathcal{R}>}{\sqrt{\alpha}} \sim \frac{3}{4}. \tag{129}$$

In the following we apply to summary of the work and description of the theoretical results.

6 Summary and outlook

We used non-minimally coupled Einstein-Dirac-Maxwell gravity model to study the formation and stability of a Dirac fermionic spherically symmetric static star. We find that the interaction parameter between the gravity and the electromagnetic fields in presence of the Dirac fields plays essential role in the formation and stabilization and size of a fermionic Dirac star. In fact for imaginary frequencies (not shown) the Dirac spinor waves suppress in time and so the fermionic matter of this kind of Dirac star decay which means that the star is unstable. In this work, we just extract the positive real frequencies of the Dirac waves and find some permissable eigenvalues of the total angular momentum of the Dirac spinors where the star remain stable. Mathematical derivations show quasi-spiral stable of the obtained solutions in phase space, because some of the eigenvalues of the Jacobi matrix are zero and some of them are complex whose real parts are negative. Hence to have stability more for such a Dirac star we must choose other kinds of the interaction terms. Recently we investigate stability of a Boson star in presence of the Coleman-Weinberg potential which is applicable in the cosmic inflation [57]. Further, more researchers show that boson fermion relativistic stars (see for instance [58], [59] and references therein) may be stable more in which bosons and fermions participate just in gravitational interactions and such a stellar objects can be suitable candidates for dark stars instead of central black holes of the galaxies. This encourages us to extent this work by considering a boson-fermion interacting matter in presence of alternative potentials.

Authors' Contributions

The authors contributed to data analysis, drafting, and revising of the paper and agreed to be responsible for all aspects of this work.

Data Availability

No data available.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or non profit sectors.

References

[1] Linde, A. D., 2008, Lect. Notes Phys. 738, 1, [arXiv:0705.0164[hep-th]]

[2] Amendola, L. & Tsujikawa, S., 2010, Dark Energy theory and observations, Cambridge University Press.

[3] Parker, L., 1971, Phys. Rev. D 3, 346.

[4] Herdeiro, C. A. R., Pombo, A. M. & Radu, E., 2017, Phys. Lett. B773, 654, [arXiv:1708.05674[gr-qc]]

[5] Kain, B., 2023, Phys. Rev. D 108, 084010.

[6] Brill, D. R., & Wheeler, J. A., 1957, Rev. Mod. Phys. 29, 465.

[7] Finster, F., Smoller J. & Yau, S. T., 1999, Phys. Rev. D 59, 104020, [arXiv:9801079[gr-qc]]

[8] Finster, F., Smoller, J., & Yau, S. T., 1999, Phys. Lett. A 259, 431, [arXiv:9802012[gr-qc]]

[9] Nodari, S. R., 2010, Ann. Henri Poincare 10, 1377.

[10] Nodari, S. R., 2010, Comptes Rendus Mathematique 348, 791.

[11] Huang, L. X., Sun, S. X., Zhang, R., Liang, C. & Wang, Y. Q., 2023, [arXiv:2309.16497[gr-qc]]

[12] Ivanenko, D. 1938, Sov. Phys. 13, 141.

[13] Weyl, H., 1950, Phys. Rev. 77, 699.

[14] Heisenberg, W., 1953, Physica 19, 897.

[15] Finkelstein, R., LeLevier R. & Ruderman, M., 1951, Phys. Rev. 83, 326.

[16] Finkelstein, R., Fronsdal C. F., & Kaus, P., 1956, Phys. Rev. 103, 1571.

[17] Soler, M., 1970, Phys. Rev. D 1, 2766.

[18] Soler, M., 1973, Phys. Rev. D 8, 3424.

[19] Ranada A. F. & Soler, M., 1973, Phys. Rev. D 8, 3430.

[20] Leith, P. E. D., Hooley, C. A., Horne K., & Dritschel, D. G., 2020, Phys. Rev. D 101, 106012, [arXiv:2002.02747[gr-qc]]

[21] Leith, P. E. D., Hooley, C. A., Horne K., & Dritschel, D. G., 2021, Phys. Rev. D 104, 046024, [arXiv:2105.12672[gr-qc]]

[22] Dzhunushaliev, V., & Folomeev, V., 2019, Phys. Rev. D 99, 084030, [arXiv:1811.07500[gr-qc]]

[23] Schunck, F. E., & Mielke, E. W., 2003, Class. Quantum Grav., 20, 301, [arXiv:0801.0307[astro-ph]]

[24] Herdeiro, C., Perapechka, I., Radu E., & Shnir, Y., 2022, Phys. Lett. B 824, 136811, [arXiv: 2111.14475[gr-qc]]

[25] Herdeiro C. A. R., & Radu, E., 2020, Symmetry 12, 2032, [arXiv:2012.03595[gr-qc]]

[26] Daka, E., Phan N. N., & Kain, B., 2019, Phys. Rev. D 100, 084042, [arXiv:1910.09415[gr-qc]]

[27] Liang, C., Sun, S. X., Ren, J. R., & Wang, Y. Q., 2024, EPJC 84, 14, [arXiv:2306.11437[hep-th]]

[28] Liang, C., Ren, J. R., Sun S. X., & Wang, Y. Q., 2023, JHEP 02, 249, [arXiv:2207.11147[gr-qc]]

[29] Herdeiro, C., Perapechka, I., Radu, E., & Shnir, Y., 2019, Phys. Lett. B 797, 134845, [arXiv:1906.05386[gr-qc]]

[30] Salcedo J. L. B., Knoll, C., & Radu, E., 2021, Phys. Rev. Lett. 126, 101102, [arXiv:2010.07317[gr-qc]]

[31] Bolokhov, S.V., Bronnikov, K.A., Krasnikov, S., & Skvortsova, M.V., 2021, Gravitation and Cosmology 27, 401, [arXiv:2104.10933[gr-qc]]

[32] Konoplya, R. A., & Zhidenko, A., 2022, Phys. Rev. Lett. 128, 091104, [arXiv:2106.05034[gr-qc]]

[33] Kain, B., 2023, Phys. Rev. D 108, 044019, [arXiv:2305.11217[gr-qc]]

[34] Rocha, R. D., 2020, Symmetry 12, 508.

[35] Minamitsuji, M„ 2020, Phys. Rev. D 102, 044048, [arXiv:2008.12758[gr-qc]]

[36] Dzhunushaliev, V., & Folomeev, V., 2020, Phys. Rev. D 101, 024023, [arXiv:1911.11614[gr-qc]]

[37] Zhang, X. Y., & Wang, Y. Q. 2024, [arXiv:2401.04547[gr-qc]]

[38] Lawrence, P., 2001, Differential equations and dynamical systems, Springer, ISBN: 964-6213-52-9.

[39] Ghaffarnejad, H. & Yaraie, E., 2017, Gen. Relativ. Gravit.49, 49, [arXiv:1604.06269[physics.gen-ph]]

[40] Ghaffarnejad, H., Ghorbani T., & Eidizadeh, F., 2023, Gen. Relativ. Gravit. 55, 135, [arXiv:2301.00682[gr-qc]]

[41] Thompson J. M. T., & Stewart, H. B., 2002, Nonlinear Dynamics and Chaos, Second Edition, John Wiley & Sons.

[42] Turner, M. S., 1988, Phys. Rev. D 37, 2743.

[43] Kain, B., 2023, Phys. Rev. D107, 124001, [arXiv:2304.10627[gr-qc]]

[44] Balakin, A. B., Bochkarev V. V. & Lemos, J. P. S., 2008, Phys. Rev. D77 ,084013, [arXiv:0712.4066[gr-qc]]

[45] Birrell, N. D. & Davies, P. C. W., 1984, Quantum Fields in Curved Space, Cambridge Univ. Press.

[46] Parker L. E., & Toms, D., 2009, Quantum Field Theory in Curved Spacetime, Cambridge Univ. Press.

[47] Carroll, S., 2004, Spacetime and Geometry, an Introduction to General Relativity, Addison-Wesley, San Francisco.

[48] Fursaev, D. & Vassilevich, D., 2011, Operators, Geometry and Quanta, Methods of Spectral Geometry in Quantum Field Theory, Springer, New York.

[49] Lord, E.A., 1976, Tensors, Relativity and Cosmology, Tata McGraw-Hill Publishing Co., Ltd., New Delhi.

[50] Alcubierre, M., 2008, Introduction to 3+1 numerical relativity, Oxford Univ. press.

[51] Finster, F., Smoller, J., & Yau, S.T., 1999, Phys. Rev. D59, 104020, [arXiv:9801079[gr-qc]]

[52] Finster, F., Smoller, J., & Yau, S.T., 1999, Phys. Lett. A 259, 431, [arXiv:9802012[gr-qc]]

[53] Wald, R. M., 1995, Quantum Field Theory in Curved Space- Time and Black Hole Thermodynamics, University of Chicago Press.

[54] Sakurai J. J. & Napolitano, J., 2011, Modern Quantum Mechanics, 2nd ed, ISBN: 978-0-8053-8291-4, Addison Wesley, (Persian languadge translated by Ghaffarnejad, H., 2017, Nopardazan Publisher, ISBN: 978-964-975-233-4).

[55] Alcubierre, M., 2025, , [arXiv:2503.03918[gr-qc]]

[56] Brihaye, Y., Ducobu L., & Hartmann, B., 2020, Phys. Let. B 811 135906.

[57] Eidizadeh F., & Ghaffarnejad, H., 2025, Nucl. Phys., B1010, 116764, [arXiv:2402.00061[gr-qc]]

[58] Mourelle, J. C., Adam, Ch., Bustillo, J. C., & Gual, N. S., 2024, [arXiv:2403.13052[gr-qc]]

[59] Alvarado, S. V., Palenzuela, C., Alic, D.,& Lopez, L. A. U., 2023, Phys. Rev. D 87, 084040, [arXiv:1210.2299[gr-qc]]