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Massive superparticles quantization in cosmological supergravity

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Abstract. In this paper, we investigate geometric quantization of massive superparticles in four dimensional space-time which preserves $\frac{1}{4}$ of the target space supersymmetry. Because the black holes (also massive black holes) are strong gravitational system, hence, application of such quantization to the supersymmetric cosmological black holes would give us information about quantum gravity. As such supermassive black holes are candidate of dark matter hence our calculations are important from cosmological point of view. The world-line action of this model contains a Wess-Zumino term which breaks $d = 4$ Lorentz symmetry. We solve Hamiltonian equation and obtain unique solution, which helps to calculate prequantization operator. It yields to corresponding Dirac equation which is important in relativistic quantum mechanics. It will be useful in the second quantization of the same models. On the other hand, superparticles are themselves candidates of dark matter, and dark matter is important for its gravitation effects. Hence, their quantization may yield to quantum gravity theory.

Keywords: Cosmological black hole; Geometric quantization; Quantum mechanics; Superparticle; Prequantization operator.

1 Introduction

As we know, some of D-branes are massive superparticles which are important in supergravity theories [1]. Geometric quantization is one of the important types of quantization, which is a procedure associates a quantum system with a given classical system [2]. Meanwhile, not every physically interesting system has $\mathcal{R}^{2n}$ as its phase space. However, it is possible to consider spin bundle and obtain prequantization operator and vector field [3]. In order to quantize more general symplectic manifolds N, one can construct a line bundle L over N along a connection ∇ on the line bundle L which has a curvature of $\omega/\hbar$. Therefore, we define prequantum operators, acting on sections of L. These operators give us useful relations between Poisson brackets and commutators [4].
In the Ref. [5] we obtained the closed 2-form of 2D black holes with pequantization method. Indeed, we considered the effective Hamiltonian of two dimensional dilatonic black hole and added axion field to calculate closed 2-form by using the geometric prequantization method. In the next step, we would like to consider a system of massive superparticle in four dimensions with $\frac{1}{4}$ supersymmetry [6], and calculate prequantization operators. Such a system is useful to gain insight of superstring theory [8]. Recently, there has been a lot of interest to study superparticles in various dimensions. However, we are interested to the

four-dimensional case. Recently, superparticle models with 3/4 or 1/4 partial breaking of global supersymmetry have been constructed [9, 10, 11, 12]. In order to have such supersymmetry, it is crucial to extend the ordinary superspace by introducing the new central charge of bosonic coordinates. By using results of the Ref. [11], quantum aspects of particular $N = 8 \to N = 2$ supersymmetric model as a typical example of massive superparticles with 1/4 partial breaking of global supersymmetry has been studied by the Ref. [6], and we use results of this work to study geometric prequantization method. Moreover, the worldline action of this model contains a Wess-Zumino term [7] which breaks $d = 4$ Lorentz symmetry. Geometric quantization is indeed an attempt to construct a quantum Hilbert space, together with appropriate operators, starting from a physical system having an arbitrary 2n-dimensional symplectic manifold as its phase space. To quantize these more general symplectic manifolds, we will use Kostant's prequantization procedure[4]. In this approach, we find a complex line bundle $L \to Q$ with a connection ∇ such that its curvature is ($i/\hbar$ times) the symplectic form ω. In the Ref. [13] the prequantization procedure in the context of symplectic super manifolds with a symplectic form which is not necessarily homogeneous has been studied. In the present work, we investigate geometric quantization for $N = 8 \to N = 2$ model as a typical example of massive superparticles with 1/4 partial breaking of global supersymmetry in an ordinary four dimensional Minkowski space-time. This paper is organized as follows. Hamilton's principal is presented in section 2. In section 3, prequantization operator of mentioned model is calculated. In section 4, we give conclusion and proposal for future works.

2 Hamilton's principle

The dynamical behavior of the system is determined by the Lagrangian $L(q, \upsilon) \in C^\infty(TQ)$, where Q is called the configuration space and TQ denotes the velocity phase space. As we know, the Lagrangian contains all the necessary information about the distribution of mass within the system and the external forces that act on it [14].
The variational equation $\delta I = 0$ is equivalent to the statement that, the orbits are solutions of the following Lagrange's equations,

$$\frac{d}{dt}(\frac{\partial L}{\partial \upsilon^a}) - \frac{\partial L}{\partial q^a} = 0, \tag{1}$$

where $\dot{q}^a = \upsilon^a$ is used as velocity. This implies that the tangent vector X to any orbit in TQ space satisfies

$$\iota_X \omega_L + dH = 0, \tag{2}$$

where ι_X is the interior multiplication, and ω_L is the closed 2-form which is corresponding to a 2-vector field on a Riemannian manifold. It is holographic dual of an incompressible flow which is given by,

$$\omega_L = \frac{\partial^2 L}{\partial q^a \partial \upsilon^b} dq^a \wedge dq^b + \frac{\partial^2 L}{\partial \upsilon^a \partial \upsilon^b} d\upsilon^a \wedge dq^b, \tag{3}$$

and H is the Hamiltonian written as follow,

$$H = \upsilon^a \frac{\partial L}{\partial \upsilon^a} - L, \tag{4}$$

which is indeed Legendre transformation of Lagrangian. Hence, we can obtain,

$$dH = \frac{\partial^2 L}{\partial \upsilon^a \partial \upsilon^b} \upsilon^a d\upsilon^a + \frac{\partial^2 L}{\partial \upsilon^a \partial q^b} \upsilon^a dq^b - \frac{\partial L}{\partial q^a} dq^a. \tag{5}$$

In the case of ω_L, this can also be concluded from $\omega_L = d\alpha_L$, where

$$\alpha_L = \frac{\partial L}{\partial v^a} dq^a, \tag{6}$$

together with the fact that the 1-form θ_L is characterized by the following coordinate-free construction.
A Lagrangian L is regular or nondegenerate whenever ω_L is everywhere nondegenerate. It means that,

$$det[\frac{\partial^2 L}{\partial v^a \partial v^b}] \neq 0, \tag{7}$$

at every point of TQ space [14]. If the Lagrangian L is regular, then ω_L is a symplectic structure on TQ and the equation (2) is equivalent to the Lagrange's equation (1). If we consider (TQ, ω_L) simply as a symplectic manifold, then there is no reason to prefer one canonical coordinate system over another. However, the Lagrangian also picks out a preferred class of canonical on TQ in which the q is coordinate on Q (pulled back to TQ) and the p is defined by,

$$p_a = \frac{\partial L}{\partial v^a}. \tag{8}$$

They are canonical since $\alpha_L = p_a dq^a$, and so $\omega_L = dp_a \wedge dq^a$. The coordinate p_a is the generalized momentum conjugate to q^a. If Lagrangian does not include q^a explicitly, then p_a is a constant.
The operator $Q_{pre}(f)$ corresponding to $f \in C^\infty(M)$ is defined as follow

$$Q_{prq}(f)s = -i\hbar(X_f - \frac{i}{\hbar}\alpha(X_f))s + fs \tag{9}$$

where s lies in some appropriate subset of Hilbert space. $Q_{prq}(f)$ is called prequantization operator (or usually called Kostant-Souriau prequantum operator), and it is always symmetric, and if X_f is complex, then it is actually self-adjoint which is Hamiltonian vector field associated to the function f. The construction of the operator $Q_{prq}(f)$ is famous as prequantization. In the next section, we give more details about prequantization operator.

3 Prequantization Operator

The action functional of a massive superparticle model exhibiting 1/4 partial breaking of global supersymmetry, is written as [6, 15],

$$S = \int d\tau \{\frac{1}{2e}(-\Pi^0\Pi^0 + \Pi^i\Pi^i\} - \frac{1}{2}em^2 + im(\theta\dot{\overline{\theta}} - \psi^i\dot{\overline{\psi}})\} \tag{10}$$

where

$$\begin{aligned} \Pi^0 &= \dot{x}^0 + \frac{i}{2}\theta\dot{\overline{\theta}} + \frac{i}{2}\overline{\theta}\dot{\theta}\frac{i}{2}\psi^i\dot{\overline{\psi}}^i + \frac{i}{2}\overline{\psi}^i\dot{\psi}^i, \\ \Pi^i &= \dot{x}^i + i\psi^i\dot{\theta} + i\overline{\psi}^i\dot{\overline{\theta}}, \end{aligned} \tag{11}$$

with $i = 1, 2, 3$, and θ, ψ^i are four complex ferminos parameterizing the odd sector of the model. Spinor θ (or ψ^i) and its complex conjugate $\overline{\theta}$ (or $\dot{\overline{\psi}}^i$) are depend on time. In that case the Lagrangian is given by,

$$L = \frac{1}{2e}(-\Pi^0\Pi^0 + \Pi^i\Pi^i\} - \frac{1}{2}em^2 + im(\theta\dot{\overline{\theta}} - \psi^i\dot{\overline{\psi}}^i). \tag{12}$$

Using the canonical momentum defined in [15], the canonical Hamiltonian H is obtained as,

$$H = \frac{1}{2}e(-P^0P^0 + P^iP^i + m^2). \tag{13}$$

We obtain 2-form defined in the equation (3) as following,

$$\begin{aligned}\omega_L &= \frac{1}{2e}(-\frac{i^2}{2}\theta\dot{\overline{\theta}} + \frac{i^2}{2}\dot{\theta}\overline{\theta})d\overline{\theta}\wedge d\theta \\ &+ \frac{1}{2e}(-\frac{i^2}{2}\psi^i\dot{\overline{\psi}}^i + \frac{i^2}{2}\dot{\psi}^i\overline{\psi}^i)d\overline{\psi}^i\wedge d\psi^i \\ &+ \frac{1}{2e}(-\frac{i^2}{2}\dot{\overline{\psi}}^i\overline{\theta} + 2i\Pi^i + 2i^2\dot{\theta}\psi^i)d\theta\wedge d\psi^i \\ &+ \frac{1}{2e}(-\frac{i^2}{2}\dot{\psi}^i\theta + 2i\Pi^i + 2i^2\dot{\overline{\theta\psi}}^i)d\overline{\theta}\wedge d\overline{\psi}^i. \end{aligned}\tag{14}$$

Since, Hessian matrix of L, which is important to study stability of the system, is not zero. Therefore ω_L is a symplectic structure and we have,

$$\begin{aligned}dH &= \frac{-1}{2e}(\frac{i^2}{2}\theta\dot{\overline{\theta}}^2)d\theta - \frac{1}{2e}(\frac{i^2}{2}\dot{\theta}^2\overline{\theta})d\overline{\theta} \\ &- \frac{1}{2e}(\frac{i^2}{2}\psi^i(\dot{\overline{\psi}}^i)^2 + 2i\dot{\theta}\Pi^i)d\psi^i - \frac{1}{2e}(\frac{i^2}{2}(\dot{\psi}^i)^2\overline{\psi}^i + 2i\dot{\overline{\theta}}\Pi^i)d\overline{\psi}^i.\end{aligned}\tag{15}$$

The second line is similar to supersymmetric prequantization operator. Then, using $\omega = d\alpha$ and the equation (6) we have

$$\begin{aligned}\alpha_L &= \frac{1}{2e}(-i\psi^i\Pi^i)d\overline{\psi}^i + \frac{1}{2e}(-i\overline{\theta}\Pi^0 + i\psi^i\Pi^i)d\theta \\ &+ \frac{1}{2e}(-i\theta\Pi^0 + 2i\overline{\psi}^i\Pi^i)d\overline{\theta} + \frac{1}{2e}(-i\overline{\psi}^i\Pi^0)d\psi^i.\end{aligned}\tag{16}$$

Therefore, we separate the equation (2) to the following equations,

$$\begin{aligned}(\frac{i^2}{2}\theta\dot{\overline{\theta}} - \frac{i^2}{2}\dot{\theta}\overline{\theta})f_1 - (-\frac{i^2}{2}\dot{\overline{\psi}}^i\overline{\theta} + 2i\Pi^i + 2i^2\dot{\theta}\psi^i)f_3 &= \frac{i^2}{2}\theta\dot{\overline{\theta}}^2 \\ (-\frac{i^2}{2}\psi^i\dot{\overline{\psi}}^i + \frac{i^2}{2}\dot{\psi}^i\overline{\psi}^i)f_4 - (-\frac{i^2}{2}\dot{\overline{\psi}}^i\overline{\theta} + 2i\Pi^i + 2i^2\dot{\theta}\psi^i)f_1 &= \frac{i^2}{2}\psi^i(\dot{\overline{\psi}}^i)^2 + 2i\dot{\theta}\Pi^i \\ (-\frac{i^2}{2}\theta\dot{\overline{\theta}} + \frac{i^2}{2}\dot{\theta}\overline{\theta})f_2 - (-\frac{i^2}{2}\dot{\psi}^i\theta + 2i\Pi^i + 2i^2\dot{\overline{\theta\psi}}^i)f_4 &= \frac{i^2}{2}\dot{\theta}^2\overline{\theta} \\ (-\frac{i^2}{2}\dot{\psi}^i\theta + 2i\Pi^i + 2i^2\dot{\overline{\theta\psi}}^i)f_2 - (-\frac{i^2}{2}\psi^i\dot{\overline{\psi}}^i + \frac{i^2}{2}\dot{\psi}^i\overline{\psi}^i)f_3 &= \frac{i^2}{2}(\dot{\psi}^i)^2\overline{\psi}^i + 2i\dot{\overline{\theta}}\Pi^i.\end{aligned}\tag{17}$$

If we consider X_L as follow,

$$X_L = f_1\frac{\partial}{\partial\theta} + f_2\frac{\partial}{\partial\overline{\theta}} + f_3\frac{\partial}{\partial\psi^i} + f_4\frac{\partial}{\partial\overline{\psi}^i}, \tag{18}$$

then, coefficients f_1, f_2, f_3, f_4 obtained from the system of equations (17).
Now, we are ready to calculate prequantization operator,

$$Q_{pre}(L)s = -i\hbar\{X_L - \frac{i}{\hbar}\theta(X_L)\}s + Ls$$

$$
\begin{aligned}
&= -i\hbar X_L s - (\frac{1}{2e}(-i\bar{\theta}\Pi^0 + i\psi^i\Pi^i)f_1 - \frac{1}{2e}(-i\theta\Pi^0 + 2i\overline{\psi}^i\Pi^i)f_2 \\
&- \frac{1}{2e}(-i\overline{\psi}^i\Pi^0)f_3 - \frac{1}{2e}(-i\psi^i\Pi^i)f_4)s \\
&+ (\frac{1}{2e}(-\Pi^0\Pi^0 + \Pi^i\Pi^i) - \frac{1}{2}em^2 + im(\theta\dot{\bar{\theta}} - \psi^i\dot{\overline{\psi}}))s, \qquad (19)
\end{aligned}
$$

where s is complex line bundle. The first term of right hand side is like momentum operator in ordinary quantum mechanics, and the last term behaves like kinetic energy.
We can also obtain prequantization operator in the momentum phase space. In that case, we have (q^i, P_i) coordinates of momentum phase space $T^*\mathcal{R}^4$. We obtain 2-form as follow,

$$
\omega_H = -edP^0 \wedge dx^0 + edP^i \wedge dx^i, \qquad (20)
$$

and therefore,

$$
dH = \frac{1}{2}e(-2P^0 dP^0 + 2P^i dP^i). \qquad (21)
$$

By using the equation $\omega = d\alpha$, we have

$$
\alpha = -eP^0 dx^0 + eP^i dx^i. \qquad (22)
$$

Then, the equation (2) implies the following unique solution,

$$
X = \frac{1}{2}P^0\frac{\partial}{\partial x^0} + \frac{1}{2}P^i\frac{\partial}{\partial x^i}. \qquad (23)
$$

It help us to calculate prequantization operator as follow,

$$
\begin{aligned}
Q_{per}(H)s &= i\hbar\{X - \frac{i}{\hbar}\alpha(X)\}s + Hs \\
&= i\hbar(\frac{1}{2}P^0\frac{\partial}{\partial x^0} + \frac{1}{2}P^i\frac{\partial}{\partial x^i})s + (\frac{1}{2}eP^0P^0 - \frac{1}{2}eP^iP^i)s \\
&+ \frac{1}{2}e(-P^0P^0 + P^iP^i + m^2)s \\
&= i\hbar(\frac{1}{2}P^0\frac{\partial}{\partial x^0} + \frac{1}{2}P^i\frac{\partial}{\partial x^i})s + m^2 s. \qquad (24)
\end{aligned}
$$

where s is a complex line bundle. In the case of $\alpha = 0$, prequantization operator is tensorial on the line bundle and it is in fact just the Dirac equation. It is the case if $p^0 dx^0 = p^i dx^i$, and hence, $Q_{per}(H) = i\hbar p^0\frac{\partial}{\partial x^0} + m^2 = -p^2 + m^2$. In presence of α, it may be modified Dirac equation including an extra term which shows Zitterbewegung effect at the Compton wavelength. However, as we show in the equation (24) the effect of α vanishes in the simplest line bundle and we have ordinary Dirac equation as the final result.

4 Conclusion

In this paper, we used a mathematical tool, so called geometric quantization, to obtain physical results, which is physically useful when the Hilbert space is too large. Therefore, we obtained prequantization operator of a massive superparticle model with 1/4 partial breaking of global supersymmetry which propagates in four dimensional flat space-time. According to the Lagrangian and Hamiltonian, the prequantization operator depends on spinor (1-form) θ. We calculated 1-form θ with $\omega = d\theta$ and obtained corresponding Dirac equation which plays central role in all relativistic quantum mechanics. In the case of $\alpha = 0$, we found prequantization operator as $-p^2 + m^2$. If we consider orthonormal frame bundle which may be considered as a bundle of Lorentzian observers, then the bundle together with the symplectic structure which interpreted as the phase space of relativistic observers have nontrivial interaction between observer and object which is important in quantum mechanical phenomenon. By using the results of this paper one can easily study mentioned progress.

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References

[1] Knutt-Wehlau, M. E., & Mann, R. B. 1998, Nucl. Phys. B, 514, 355

[2] Geraci, J., [arXiv:0911.0964 [math.SG]]

[3] Fulp, R. O., Lawson, J. K., & Norris, L. K. 1994, Int. J. Theor. Phys., 33, 1011

[4] Hall, B. C., 2013, Springer Science

[5] Taleshian, A., Shaban Nataj, M., & Pourhassan, B. 2014, Int. J. Theor. Phys., 53, 3943

[6] Bellucci, S., Galajinsky, A., Ivanov, E., & Krivonos, S. 2002, Phys. Rev. D, 65, 104023

[7] Sadeghi, J., Banijamali, A., & Pourhassan, B. 2007, Acta Phys. Pol. B, 38, 3143

[8] Green, M. B., & Schwarz, J. H. 1984, Phys. Lett. B, 136, 367

[9] Bandos, I., & Lukierski, J. 1999, Mod. Phys. Lett. A, 14, 1257

[10] Bandos, I., Lukierski, J., & Sorokin, D. 2000, Phys. Rev. D, 61, 045002

[11] Delduc, F., Ivanov, E., & Krivonos, S. 2000, Nucl. Phys. B, 576, 196

[12] Fedoruk, S., & Zima, V. 2000, Mod. Phys. Lett. A, 15, 2281

[13] Tuynman, G. M. 2010, J. Geom. Phys., 60, 1919

[14] Woodhouse, N. M. J. 1991, Geometric Quantization, Oxford University Press

[15] Farahat, N. I., & Nassar, Z. 2013, J. App. Math. Phys., 1, 105