Iranian Journal of Astronomy and Astrophysics
Vol. 4, No. 3, Autumn 2017
©Available online at http://journals.du.ac.ir
DOI:10.22128/ijaa.2017.124

Iranian Journal of Astronomy and Astrophysics

Introducing Stable Real Non-Topological Solitary Wave Solutions in 1+1 Dimensions

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Abstract. By adding a proper term to the Lagrangian density of a real non-linear KG system with a proposed non-topological unstable solitary wave solution, its stability guaranteed appreciably. This additional term in the new modified Lagrangian density behaves like a strong internal force which stands against any arbitrary small deformation in the proposed non-topological solitary wave solution.

Keywords: non-linear KG systems, non-topological, solitary wave solutions, stability, soliton.

1 Introduction

The Real non-linear KG systems in $1+1$ dimensions with kink (anti-kink) bearing solutions were studied for decades [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Kink (anti-kink) solutions are topological objects. Topological characteristics are the main reason for their stability. Mathematically, a solitary wave solution is stable if its rest energy would be a minimum among other solutions which are close to the solitary wave solution. In other words, for a stable solitary wave solution, any small permissible deformation leads to an increase in the rest energy. However, the non-topological solutions are more interesting since without any consideration, just by adding them together when they are far enough from each other, it leads to a solution again, i.e. many particle-like solution can be constructed easily. Many efforts have been made to find non-topological relativistic Klein Gordon solitary wave solutions among which one can mention the complex non-linear KG systems which yield the non-topological solitary wave-packet solutions [20, 21, 22, 23, 24, 25, 26, 27], and Wazwaz's works, but none of them are stable objects (like kink solutions) as solitons mathematically [28, 29].

In this paper, we restrict ourselves to the real relativistic non-linear Klein-Gordon field systems in $1+1$ dimensions with non-topological solitary wave solutions. Our main goal is to show how adding a new proper self interaction term to the Lagrangian density of a real non-linear KG system with an unstable non-topological solitary wave solution turns the solution to a stable one so that the related known KG-like equation of motion stays unchanged. This additional term behaves like a strong internal force which stands against any arbitrary small deformations on the solitary wave solution. In this new model, all equations and relations for the non-topological stable solution are exactly the same as original ones of the real KG system. Morover, there is a constant A, which larger values of it lead to stronger stability. The original idea for introducing this new model is started from Derrick's paper and its suggestion for finding a stable solitary wave solution from a real non-linear scalar filed in $3+1$ dimensions [30], although this work is essentially different.

The organization of this paper is as follows: In the next section, we will introduce a special non-linear real KG system with a non-topological Gaussian solitary wave solution. We will use it in section III as a good simple example to show numerically how adding a proper term to the original Lagrangian density leads to stability of the non-topological solitary wave solution. The last section is devoted to summary and conclusions.

2 real non-topological Gaussian solitary wave solutions

The real non-linear KG systems in 1 + 1 dimensions can be generally introduced by a Lagrangian density in the following form:

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi - U(\varphi) = \dot{\varphi}^2 - \varphi'^2 - U(\varphi), \tag{1}$$

where prime and dot represent space and time derivatives, respectively. φ is a real scalar field and $U(\varphi)$ is called potential. The equation of motion corresponding to this Lagrangian density reads

$$\ddot{\varphi} - \varphi'' + \frac{1}{2}\frac{dU}{d\varphi} = 0. \tag{2}$$

In the whole paper, we take the speed of light equal to one ($c = 1$). The energy-momentum tensor corresponding to the Lagrangian density (1) can be calculated using the Noethers theorem:

$$T^{\mu\nu} = 2\partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}, \tag{3}$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor. Also, the related energy density has the following form:

$$T^{00} = \varepsilon(x,t) = \dot{\varphi}^2 + \varphi'^2 + U(\varphi) \tag{4}$$

If the potential $U(\varphi)$ has at least two degenerate vacua (i.e. minimum points of potential), there will be localized solutions called kinks and anti-kinks with positive and negative topological charges, respectively.

In the soliton paradigm, it is common firstly to choose a potential and then, search for the possible solitary wave solutions at the second step. In this paper, we will do the reverse, i.e. instead of finding solitary wave solutions from a known potential $U(\varphi)$, we are going to find a potential $U(\varphi)$ leading to a type of the proposed solitary wave solutions. For example, the proposed relativistic solitary wave solution can be a localized function in the following form:

$$\varphi(x,t) = \exp\left(-\gamma^2(x - vt)^2\right), \tag{5}$$

in which $\gamma = 1/\sqrt{1-v^2}$ and v is the velocity (Fig. 1). Using the equation of motion (2), it is easy to prove that the right potential $U(\varphi)$ which leads to the proposed solution (5) is

$$U(\phi) = -4\varphi^2\ln(\varphi), \tag{6}$$

that is plotted in Fig. 2. This potential for $\varphi > e^{-1/2}$ is continuously decreasing and for $\varphi > 1$ takes negative values. Therefore, without violating the conservation of energy, the effect of any small perturbation, leads φ to increase infinitely, i.e. the proposed solution (5) is not essentially stable and spontaneously blows up (see Fig. 3). We examined a lot of different non-topological functions instead of exp function (5) to find a proper potential $U(\varphi)$, but for all of them, we could not find a proper one. It is easy to show that generally for the real non-linear KG systems, the stable non-topological and non-vibrational solitary wave (soliton-like) solution does not exist at all.

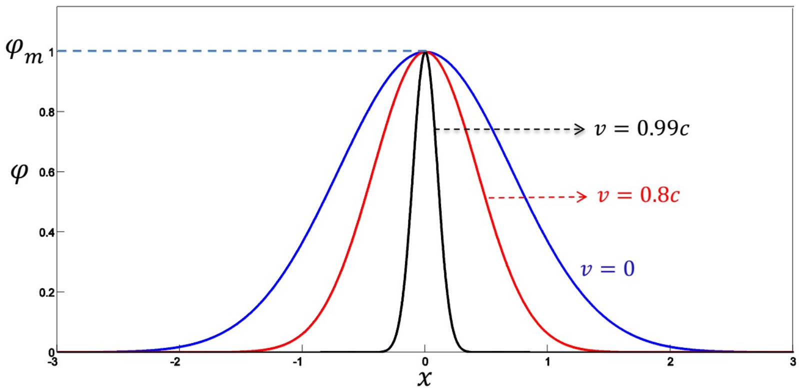

Figure 1: Localized wave function (5) at different velocities and $t = 0$ is shown. For the other relativistic localized solitary wave solutions, the same results would be obtained. Note that, their widths decrease according to Lorentz contraction.

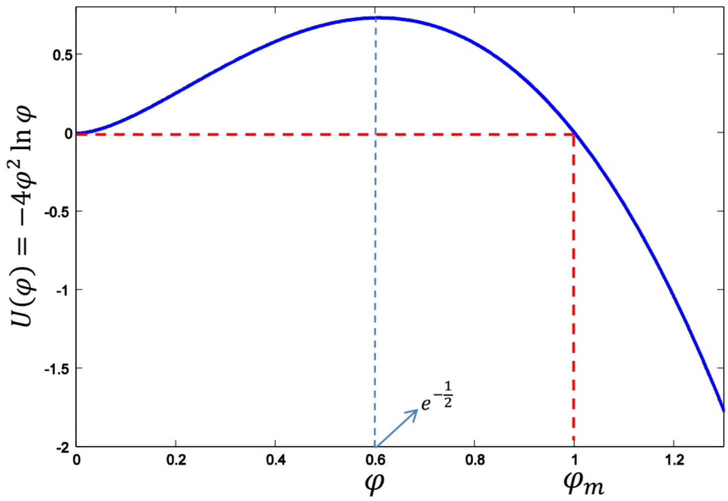

Figure 2: Plot of potential $U = -4\varphi^2 \ln(\varphi)$ versus φ.

Generally, a non-vibrational solitary wave solution is represented by

$$\varphi(x,t) = f(\widetilde{x}) = f(\gamma(x - vt)), \tag{7}$$

where $\widetilde{x} = \gamma(x - vt)$ and $f(\widetilde{x})$ can be any arbitrary well defined continuous finite function including topological and non-topological functions. For such solutions (7), the equation of motion (2) reduces to

$$\frac{d^2\varphi}{d\widetilde{x}^2} = \frac{1}{2}\frac{dU}{d\varphi}. \tag{8}$$

If we multiply it by $\frac{d\varphi}{d\widetilde{x}}$ and integrate, it turns to

$$-\partial_\mu\varphi\partial^\mu\varphi = \left(\frac{d\varphi}{d\widetilde{x}}\right)^2 = U(\varphi) + C = U(\varphi), \tag{9}$$

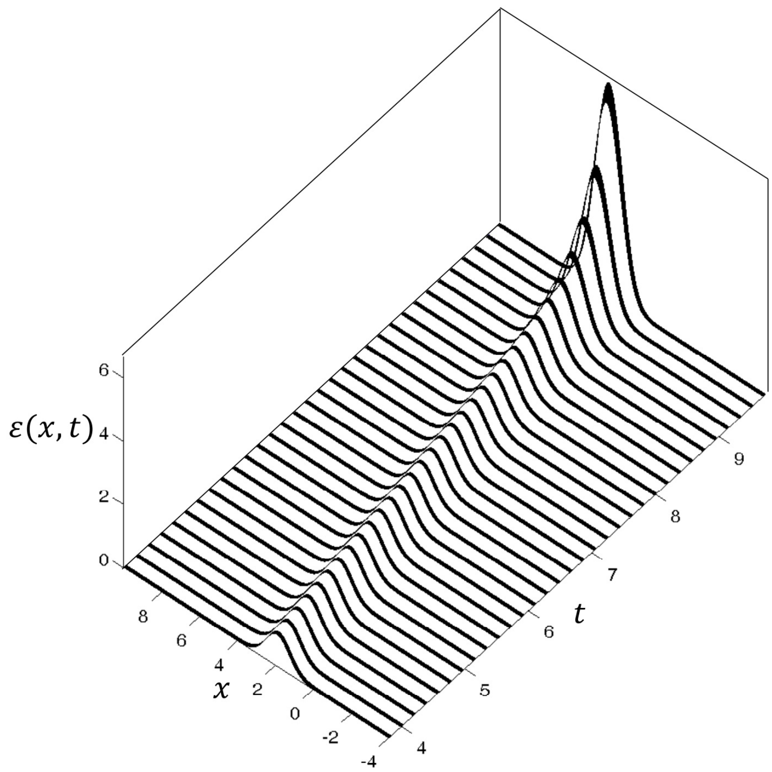

Figure 3: The non-topological and non-vibrational solutions of the real non-linear KG system are essentially unstable. This is a moving unstable solitary wave solution (5) which moves initially at $0.5c$ speed.

in which C is the integration constant and we have to take it zero to have a solitary wave solution with finite rest energy. It was shown that generally if the potential $U(\varphi)$ has n degenerate minimum or vacuum points (i.e. the special fields for which the potential $U(\varphi)$ and its first derivative $\frac{dU}{d\varphi}$, all are zero), there are $2(n-1)$ types of the topological solutions which called kinks and anti-kinks [1].

In general, to have a non-topological non-vibrational solitary wave solution with a localized function $f(\widetilde{x})$ for which $f(\pm\infty) = 0$, according to Eq. (9), there must be just a single vacuum point at $\varphi = 0$. Note that, as we said before, the existence of more than one successive vacuum points leads to kink (antikink) solutions. However, for any non-topological localized solitary wave solution such as (5), the field function $\varphi(\widetilde{x})$ changes from zero at $\widetilde{x} = \pm\infty$ to a maximum value φ_m at $\widetilde{x} = 0$ for which $\frac{d\varphi}{d\widetilde{x}}(\widetilde{x} = 0) = 0$. Hence, according to Eq. (9), φ_m is the maximum value of the non-topological solution for which $U(\varphi_m) = 0$, but $\frac{dU}{d\varphi}|_{\varphi_m}$ is not zero anymore. In other words, the field potentials $U(\varphi)$ which lead to non-topological non-vibrational solitary wave solutions must be positive in the interval $0 \leq \varphi \leq \varphi_m$, but their curves pass from φ_m to negative regions such as what we can see in the Fig. 2. Therefore, it is not possible to have a relativistic Klein-Gordon system with a stable non-topological non-vibrational solitary wave solution at all.

3 New extended nonlinear KG system with stable non-topological solitary wave solution

In the real word, stable fundamental particles exist like localized energy density lumps in the space. They can move with any arbitrary velocity. On the other hand, many of the fundamental equations which are used in the quantum field theory to consider the behavior of fundamental particles are KG-like or something like the nonlinear KG (nKG) models. Namely, the Dirac equation for electrons and positrons and non-linear ϕ^4 theory for Higgs particles are good examples which confirm this idea. Classically, we would like to consider particles as some stable solutions of the non-linear relativistic equations which are called solitary wave solutions or solitons. Stability is the main condition for a solitary wave solution to be soliton. Moreover, comparing with dynamical equation in standard quantum field theory since any KG (or nKG)-like equation was used just for a special type of fundamental particle (spin-0 particles), we would like the standard classical nonlinear KG-like dynamical equation (2) just to be satisfied for a special type of solitary wave solution (like the proposed solution 5).

All these demands of being united in a classical non-linear relativistic model with non-topological soliton solutions is a very hard and complicated request. However, one can overcome this problem via introducing an idea which can be called internal force, i.e. the force which makes strong restrictions on the structure of a solitary wave solution and makes it to be a stable solitary wave solution or soliton. In fact, we would like to find an extended nKG Lagrangian density with an additional self-interaction term which finally for a special solitary wave solution (SSWS) (5), shrinks to the same original dynamical nonlinear KG equation (2). Moreover, we want this new additional term to behave like an internal force which restricts the proposed solitary wave solution to be a stable solution. In fact, mathematically, the new extended nKG Lagrangian density is a completely different system which has different solutions, but for one of these solutions, all new relations and equations completely shrink to the same original nKG system. Moreover, we will show that for all arbitrary permissible small deformations (variations) for this special solution, the total energy always increases i.e. it is also a stable solution.

Now, to clarify how a new extended nKG system can leads to a stable solitary wave solution, we specially focus on the pervious nKG system (1) with the unstable solitary wave solution (5). At the first step, we are going to add a proper self interaction term to the original Lagrangian density (1) for which the same original solitary wave solution (5) would be a solution as well and just for this proposed solution (5), the dynamical equation shrinks to the same original one (2). However, the extended lagrangian density may be introduced in the following form:

$$\mathcal{L}_N = \partial_\mu\varphi\partial^\mu\varphi - U(\varphi) + A\mathcal{K}^3, \tag{10}$$

in which A is an arbitrary constant for dimensional reasons and $\mathcal{K}$ is a functional scalar:

$$\mathcal{K} = \partial_\mu\varphi\partial^\mu\varphi + U(\varphi), \tag{11}$$

which is inspired by equation (9). It is easy to show that for the proposed solitary wave solution (5), $\mathcal{K}$ is equal to zero. The new equation of motion is

$$\ddot{\varphi} - \varphi'' + \frac{1}{2}\frac{dU}{d\varphi} + \left[3A\mathcal{K}\frac{\partial\mathcal{K}}{\partial x^\mu}\frac{\partial\mathcal{K}}{\partial(\partial_\mu\varphi)}\right] + \left[\frac{3}{2}A\mathcal{K}^2\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{K}}{\partial(\partial_\mu\varphi)}\right)\right] - \left[\frac{3}{2}A\mathcal{K}^2\frac{\partial\mathcal{K}}{\partial\varphi}\right] = 0. \tag{12}$$

Again, it is easy to see that the proposed solitary wave solution (5) satisfies this complicated equation automatically. In fact, the three additional terms in equation (12) which are categorized in $[\cdots]$'s and contain $\mathcal{K}$ or $\mathcal{K}^2$, all are equal to zero for the proposed function (5). Therefore, for the special solution (5), the new complicated equation of motion (12) reduces to the same original one (2) as we desired. Now, the new energy density function is

$$\varepsilon(x,t) = \left[\dot{\varphi}^2 + \varphi'^2 + U(\varphi)\right] + \left[6A\dot{\varphi}^2\mathcal{K}^2 - A\mathcal{K}^3\right] = \varepsilon_o + \varepsilon_1, \tag{13}$$

which is divided into two distinct parts ε_o and ε_1. It is easy to see that for the proposed solution (5), the additional term ε_1 is equal to zero; then, the related energy density function and total rest energy (E_0) for the SSWS (5) remain unchanged.

This new extended nKG system (10) is too complicated to consider its stability via numerical simulations. To show that the stability of the solitary wave solution, we need to introduce a new criterion. In fact, each dynamical equation specifies the permissible ways which a system can be evolved related to different initial conditions. For example, for the pervious dynamical nKG equation (2) and its unstable solitary wave solution (5), according to Fig. 3, we can see many permissible deformations at different times. Therefore, for any dynamical equation with a special solitary wave solution, the possible deformations are not arbitrary or random. If for a solitary wave solution, the possible permissible small deformations (do not) need external energies, it is a stable (unstable) object which is called a soliton solution. For an unstable solitary wave solution, it can evolve to new deformations with no need to increase in total energy and hence can be occur spontaneously (as we see in Fig. 3). However, for many complicated systems, like one which we introduced in this paper (10), we are not hopeful analytically or numerically to study permissible deformations and argue about stability of their solutions. Therefore, to be sure that a solitary wave solution of a complicated system studied under all permissible deformations (which for the new extended nKG system are impossible or complicated to recognized), we study all possible arbitrary small deformations (either permissible or impermissible). If one can show that for all arbitrary small deformations, the related total energy always increase, we can be sure that all permissible deformations would be considered automatically and therefore, the solitary wave solution would be a stable one, i.e. the new criterion is satisfied.

Now, let us to concentrate specially on the second part of the energy density ε_1:

$$\varepsilon_1(x,t) = A\mathcal{K}^2(6\dot{\varphi}^2 - \mathcal{K}). \tag{14}$$

The main reason for the stability of the proposed solitary wave solution (5) is related to this relation. Since the new extended nKG model is completely relativistic and covariant, if a solution is stable in one reference frame, it would be stable in any other frames. It is easy to show that for any arbitrary (permissible or impermissible) small deformation with respect to an inertial frame which sees the solitary wave solution (5) moves at speeds near to the speed of light, $\varepsilon_1(x,t)$ is always positive and increases strongly for large values of coefficient A. To prove this result in general, an arbitrary small variation of the field function of a moving solitary wave solution is introduced in the following form:

$$\varphi(x,t) = \varphi_v(x,t) + \delta\varphi(x,t). \tag{15}$$

in which $\varphi_v(x,t) = \exp\left(-\gamma^2(x-vt)^2\right)$ is the moving solitary wave solution (5), and $\delta\varphi(x,t)$ (small deformations) can be any non-trivial arbitrary small function of space-time. Note that, a trivial small deformation is related to change of speed v to a smaller value $v-\delta v$ (i.e. another solution with a smaller speed $\varphi(x,t) = \varphi_{v-\delta v}(x,t) \approx \varphi_v(x,t) - \delta v\frac{\partial\varphi_v}{\partial v}$) for which the second part of the energy density, i.e. ε_1, remains zero and the total energy decreases. If we insert a non-trivial deformed solitary wave function (15) into $\mathcal{K}$ and keep the terms with the same order of $\delta\varphi$, it yields

$$\mathcal{K} = \mathcal{K}_v + \delta\mathcal{K} = \delta\mathcal{K} \approx \left[2\dot{\varphi}_v(\delta\dot{\varphi}) - 2\varphi'_v(\delta\varphi') + \delta\varphi\frac{dU(\varphi_v)}{d\varphi_v}\right], \tag{16}$$

where $\mathcal{K}_v = \left[(\dot{\varphi}_v)^2 - (\varphi_v')^2 + U(\varphi_v)\right]$ is written for the moving undeformed solitary wave solution (5) which essentially is zero. Now, we can calculate the second part of the energy density for the deformed solitary wave (15):

$$\begin{aligned}\varepsilon_1(x,t) = \delta\varepsilon_1 &= A(\mathcal{K}_v + \delta\mathcal{K})^2[6(\dot{\varphi}_v + \delta\dot{\varphi})^2 - (\mathcal{K}_v + \delta\mathcal{K})] - A\mathcal{K}_v^2[6\dot{\varphi}_v^2 - \mathcal{K}_v] \\ &\approx 6A(\delta\mathcal{K})^2(\dot{\varphi}_v)^2 \geqslant 0,\end{aligned} \tag{17}$$

which is always positive definite for all possible small (permissible or impermissible) deformations. Since for a moving solitary wave solution that moves at a speed near to the speed of light, $\dot{\varphi}_v$ at different space-times takes nonzero remarkable values; therefore we can approximate $(\dot{\varphi}_v + \delta\dot{\varphi})^2$ to $(\dot{\varphi}_v)^2$. If we consider a non-moving solitary wave solution, i.e. $v = 0$, the related time derivatives is zero ($\dot{\varphi}_0 = 0$) and then $\delta\varepsilon_1 \approx -A(\delta\mathcal{K})^3$ which is not positive definite anymore! Since $\delta\varepsilon_1$ is a function of cube of $(\delta\mathcal{K})$ with a unknown sing, for this case ($v = 0$), we can not say it is a positive or negative definite function. However, as we said before, since all inertial frames are equivalent, we can consider the small deformations for a special frame which see the SSWS (7) moves with a non-zero speed and non-zero remarkable values of $\dot{\varphi}_v$ in such a way that one can approximates $(\dot{\varphi}_v + \delta\dot{\varphi})^2$ to $(\dot{\varphi}_v)^2$.

Unlike ε_1, the first part of the energy density (ε_o) and total energy, may be decreasing under some arbitrary small variations of φ. But, we will show that this small decreasing behaviour, for large values of A, would be unimportant physically and can be ignored in the stability considerations. To prove this result in general, let us insert a deformed solitary wave function like (15) into $\varepsilon_o(x,t)$ and keep the terms with the same order of $\delta\varphi$, then it yields

$$\varepsilon_o(x,t) = \varepsilon_{os}(\widetilde{x}) + \delta\varepsilon_o(x,t) \approx \left[(\dot{\varphi}_v)^2 + (\varphi_v')^2 + U(\varphi_v)\right] + \left[2\dot{\varphi}_v(\delta\dot{\varphi}) + 2\varphi_v'(\delta\varphi') + \delta\varphi\frac{dU(\varphi_v)}{d\varphi_v}\right], \tag{18}$$

where $\varepsilon_{os}(\widetilde{x}) = \varepsilon_{os}(\gamma(x - vt))$ is the energy density of a moving undeformed SSWS (5). Using large value of the coefficient A, the comparison between $\delta\varepsilon_o$ and $\delta\varepsilon_1$ needs more consideration. Note that $\delta\varepsilon_o$ is of order $\delta\varphi$ and may take positive or negative values for arbitrary deformations. Instead, since $\delta\mathcal{K}$ is of order $(\delta\varphi)$, $\delta\varepsilon_1$ is of order $A(\delta\varphi)^2$. For example, consider $A \sim 10^{40}$; therefore, the order of magnitude of variations for which the solitary wave solution (5) is not mathematically a stable object, are less than 10^{-20} which physically so small that can be discarded in the stability considerations. These variations $|\delta\varphi|$ are small enough for which $|\delta\varphi| < \delta\varphi$ or $|\delta\varepsilon_o| < A(\delta\varepsilon_1)^2$. Therefore, for such small variations, the total energy E may be reduced with very small amounts equal to the integration of $\delta\varepsilon_o$ over all of the space which again is a very small unimportant value. Hence, physically if we consider large enough value for constant A, the stability of the solitary solution (5) would be effectively increased. In fact, this so small decreasing behaviour related to this fact that for a non-deformed (or for a very small deformed) SSWS (5), the dominant equation of motion is the shrunk version of the equation of motion (12) which is the same original nKG equation (2).

For example, let us consider an arbitrary deformation for a moving solitary wave solution with velocity $v = 0.5$ in the following form:

$$\varphi(x,t) = (1+\xi)\exp(-\gamma^2(x - vt)^2), \tag{19}$$

in which $\delta\varphi = \xi\exp(-\gamma^2(x-vt)^2)$ and ξ is just a small parameter. If we plot the total energy E versus ξ, the output result is shown in Fig. 4. As we can see in Fig. 4, for $A \sim 10^5, 10^6$, $E(\xi = 0) = E_v = 2.894405018233 = E_o\gamma$ is not a minimum, but by increasing A, this behavior disappears slowly i.e. $E(\xi = 0)$, when we consider large values of A, according to

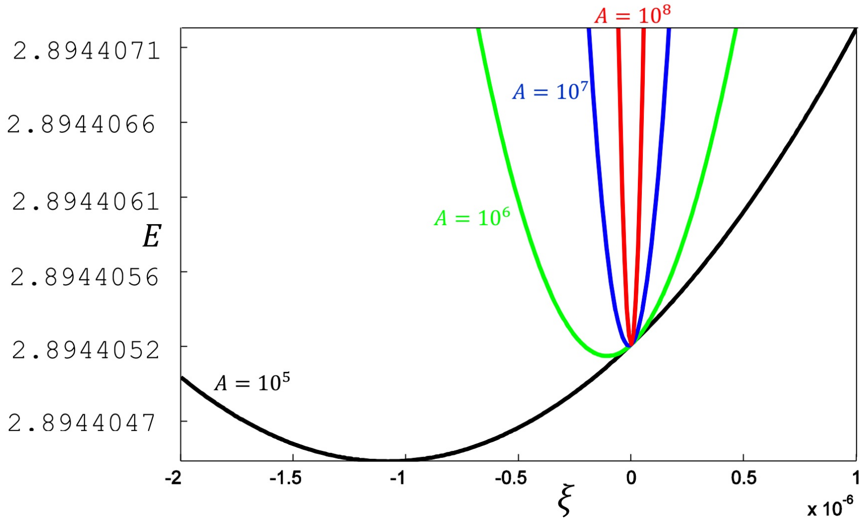

Figure 4: Variations of the total energy E versus small ξ for different A's. We have considered a moving solitary wave solution with $0.5c$ speed which deformed according to relation (19)

Fig. 4, it is apparently a minimum. If we zoom on the case $A = 10^8$ around the $\xi = 0$, the output result can be see in the Fig. 5. As we see in Fig. 5, for smaller $|\xi|$, $E(\xi = 0)$ is not really a minimum of the case $A = 10^8$. Again, by increasing A, this behavior disappears slowly i.e. we can always find a very small range for the coefficients ξ around $\xi = 0$, which $E(\xi = 0)$ is not a minimum. This range for larger A is apparently smaller. Therefore, mathematically, by using the new criterion, the special solitary wave solution (5) is not really a stable object, but physically, if we consider extremely large values of A, this is a very small shift from $E(\xi = 0) = E_v$ which is completely unimportant and the stability of the solitary wave-packet solutions is enhanced appreciably. Therefore, to a very good approximation, we can consider the single solitary wave packet solution as a stable object.

For more support, let us consider the variation of total energy for the following arbitrary one parameter deformations of the moving SSWS (5) at $t = 0$ and $v = 0.5$:

$$\varphi(x,t) = (1 + \xi t)\exp(-\gamma^2(x - vt)^2), \tag{20}$$

$$\varphi(x,t) = \exp(-(\gamma^2 + \xi x t)(x - vt)^2), \tag{21}$$

and

$$\varphi(x,t) = \exp(-\gamma^2(x - vt)^2)(1 + \frac{\xi}{(x - vt)^2 + 4}), \tag{22}$$

where ξ is just a small parameter. If we plot the total energy E versus ξ, the output results are shown in Figs. 6-8 respectively. It confirms that again how large values of A leads to more stability of the SSWS (5).

A special impermissible deformation for the moving solitary wave solution of the new extended Lagrangian density (10) can be introduced from ones which appear in Fig. 3 as permissible deformations of the original Lagrangian-density (1). According to Fig. 3, for time interval $0 < t < 8$, the shape of the Gaussian solitary wave solutions does not change remarkably. In fact, it changes so small which can not be seen in Fig. 3. If we calculate

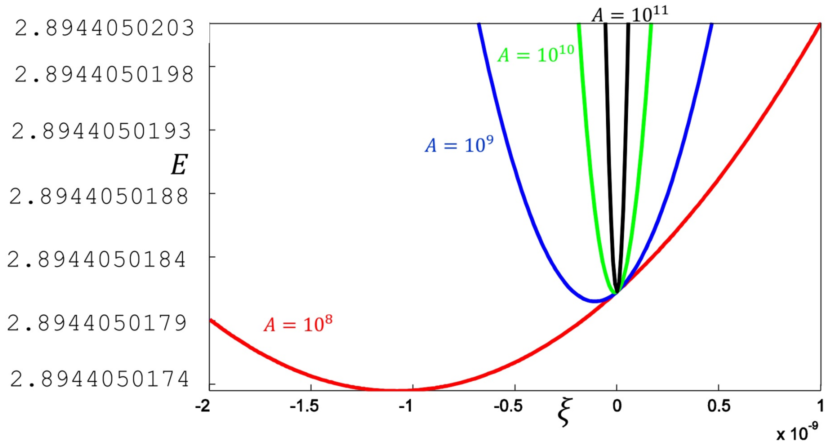

Figure 5: Variations of the total energy E versus small ξ (in compared with Fig. 4) for different A's. We have considered a moving solitary wave solution with $0.5c$ speed which deformed according to relation (19).

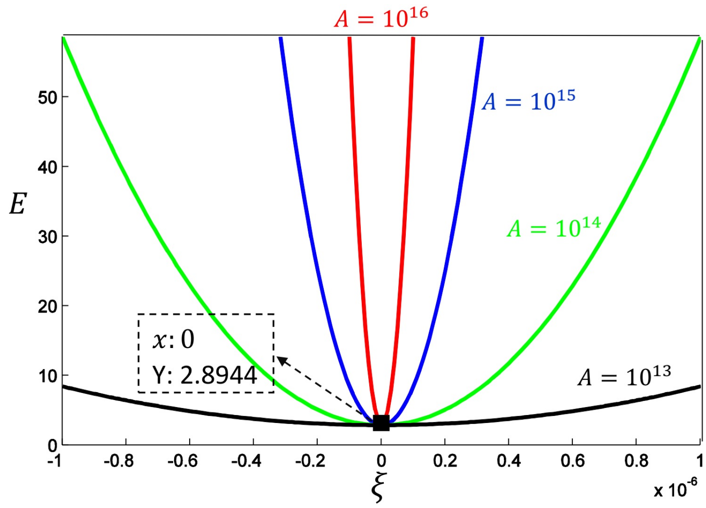

Figure 6: Variations of the total energy E versus small ξ for different A's at $t = 0$. We have considered a moving solitary wave solution with $0.5c$ speed which is deformed according to relation (20).

the total energy of the small disturbed proposed solitary wave solution (5) at different times $0 < t < 8$ (according to Fig. 3) for the original Lagrangian system (1), and plot it versus time, the output is the expected horizontal dash-red line in Fig. 9, i.e. it remains unchanged for all times and is equal to total energy (E_v) of the moving undeformed solitary wave solution (5). Now, for the small deformed profiles which appear in Fig. 3 and for time interval $0 < t < 8$, let us consider the new energy density (13) to calculate the total energy. Note that, the deformed profiles in Fig. 3 are not solutions of the new extended nonlinear KG system (10), therefore; they are impermissible deformations for the new system. However, for different A's, the new output blue curves in Fig. 9 would be resulted for such impermissible

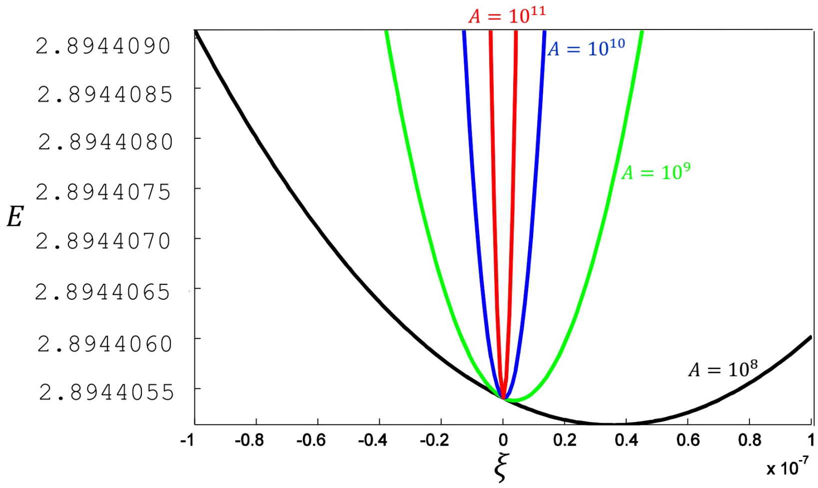

Figure 7: Variations of the total energy E versus small ξ for different A's at $t = 0$. We have considered a moving solitary wave solution with $0.5c$ speed which is deformed according to relation (21).

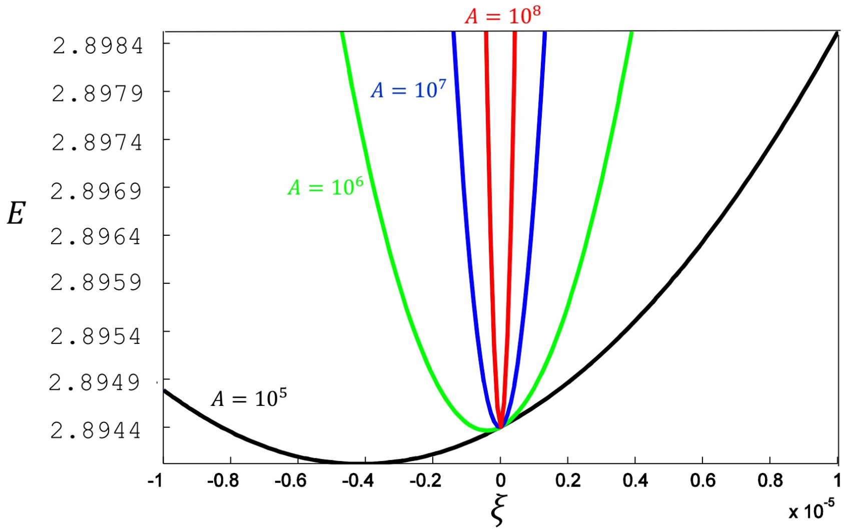

Figure 8: Variations of the total energy E versus small ξ for different A's at $t = 0$. We have considered a moving solitary wave solution with $0.5c$ speed which is deformed according to relation (22).

deformations as we expect. It again, confirms that for arbitrary small deformations of the solitary wave solution (5), the larger values of A lead to greater increase in the total energy i.e. they lead to more stability.

Briefly, since it is not possible to recognize all permissable deformations of a special solitary wave solution, we study the behaviour of its total energy for all arbitrary (impermissible or permissible) deformations. If just for one of these arbitrary deformations, there was a decreasing behaviour, the certainty about the stability is completely destroyed. Fortunately, it was shown that theoretically, for large values of A, it is not possible to find a small arbitrary variation which appreciably leads to a decreasing behaviour in the total energy (c.f. Eq. (17)). Note that the stability for a non-moving SSWS (5), for which $\dot{\varphi}_o = 0$, is somewhat complicated. Accordingly, in this paper, since relativistically there is no essential difference between a moving and non-moving solitary wave solution, we specially used a moving solitary wave solution with $v = 0.5$ to consider stability.

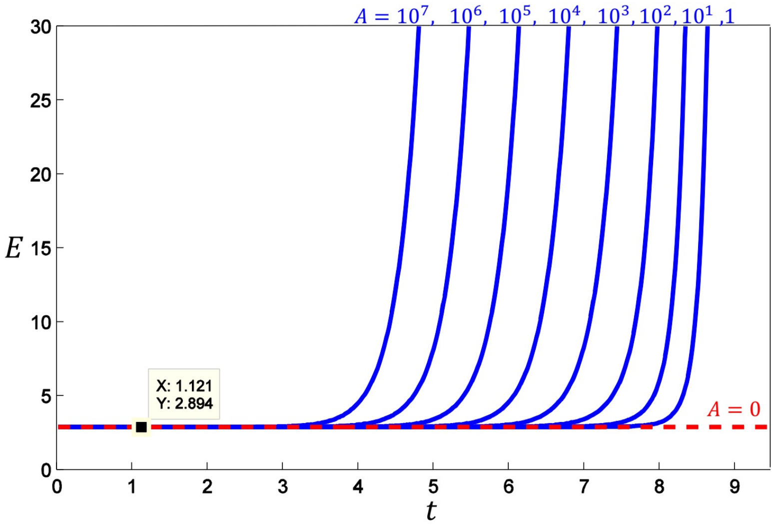

Figure 9: The total energy E versus time for all profiles which were shown in Fig. 3. The red dash-line is related to the original system (1), and the blue curves are related to our new model (10) for different A's.

It may be possible to consider some variations for which $\mathcal{K} = 0$ and $\varepsilon_1 = 0$. These possible variations impose a constraint on φ and its derivatives in the following form:

$$\dot{\varphi}^2 = \varphi'^2 - U(\varphi). \tag{23}$$

If we substitute $\dot{\varphi}^2$ from (23) in the energy density function (13), it shrinks to

$$\varepsilon(x,t) = 2\varphi'^2, \tag{24}$$

i.e. it leads the elimination of the potential term $U(\varphi)$ in the energy density function and we are not worried about the existence of any continuously decreasing potential anymore. Therefore, for these special variations, we have just one positive definite term in the first part of the energy density function (ε_o) which again is the confirmation of the stability of the non-topological solitary wave solutions (7) while $\mathcal{K} = 0$.

In general, since the SSWS (5) is a non-topological object, we can construct any arbitrary multi lumps solution just by adding SSWS's (5) together at different velocities, provided that they are far enough from each other. Figure 10 shows properly how adding many single solitary wave solutions together leads to distinct energy density lumps as many as distinct particle like objects. It confirms that the new A-dependent terms of the new system (10) for any number of the distinct solitary wave solutions (5) are approximately zero.

One may think that this strange and complicated system is not a physical case. For example, the new equation of motion (12) has some complicated terms which have not seen in other known physical relativistic (classical or quantum mechanical) field systems. But

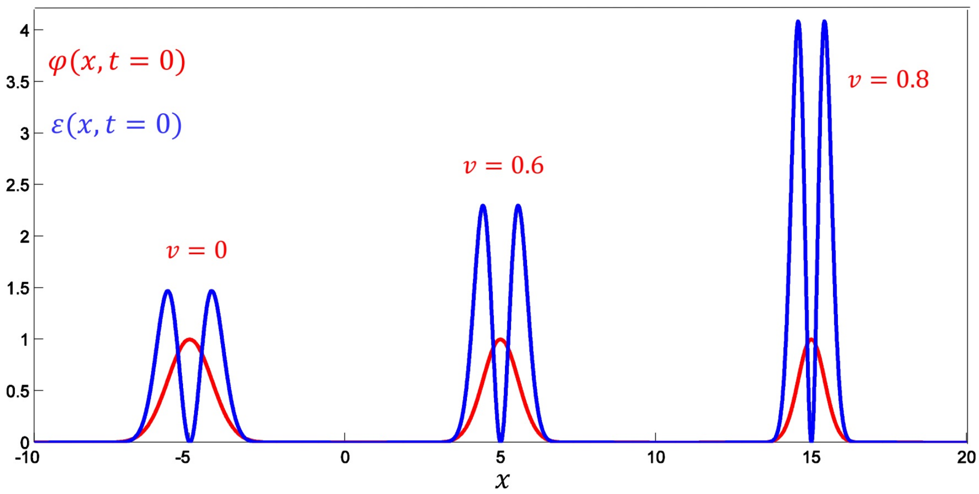

Figure 10: The field (i.e. the red curve) and the energy density (i.e. the blue curve) representations of a three lumps solution at different velocities for the case $A = 10^{16}$. These figures are exactly the same as those which can be obtained for the case $A = 0$.

note that this complicated equation (12), for the proposed solitary wave solution (5), is reduced to the original known non-linear KG-like version (2). In other words, if one asks about the equation of motion of a free stable solitary wave solution (5), our answer is the same original equation (2). The $\mathcal{K}$ dependent terms in the new equation of motion (12) will be important just when a SSWS (5) enters to a collision process (which is not considered as a free particle). In fact, we showed how a nKG-like equation (2), without the slightest change, can remain unchanged just for a special type of solutions (5) so that their stability are guaranteed automatically. Note that, for the new extended system (10), there are other solutions, but all of them have energies larger than the proposed solitary wave solution (5) (except the solutions which are very close to the vacuum state).

From quantum field theory point of view, any known KG (or non-linear KG)- like equation is applied to a special type of known real particles (for example ϕ^4 theory is applied just for Higss particles), hence classically, in this paper we showed that how a nKG-like equation (2) can be applied just for a special type of non-topological stable particle-like solutions (5) among the other possible solutions of the complicated equation (12). Moreover, quantum mechanically, we do not allowed to speak about what really happens to the particle during a collision processes, in fact, we just allowed to speak about the before and after collisions when the particle is free and its related equation of motion is the same known KG (or nKG)-like equations. Classically, for this extended model (10), we showed that when stable particles (5) are free, the dominant equation of motion is the same nKG-like equation (2), but when they enter to a collision process, the new complicated equation of motion (12), without any change, must be considered as the right equation of motion. In fact, this new extended model is a mathematical effort to show that it may be possible to consider fundamental particles as non-topological localized stable solitary wave solutions with dominant KG (or nKG)-like equations. Moreover, it was introduced specially in $1 + 1$ dimensions, but it can be similarly generalized to $3 + 1$ dimensions and other systems. The free parameter A which appears in this model is an indicator constant for the power of the stability of the solitary wave solution.

4 Summery and conclusion

For a real nonlinear relativistic Klein-Gordon field system with an unstable solitary wave solution in 1+1 dimensions, adding a proper term to the original Lagrangian density, one can build a new extended real non-linear KG (nKG) system with the same original solitary wave solution the stability of which is guaranteed appreciably. In fact, this new additional term behaves like a strong internal force which suppresses the proposed solitary wave solutions to stay in the same original shape. This additional term for the solitary wave solution is always zero and therefore, the dominant dynamical equation shrinks to the same original nKG system exactly like a real fundamental particle.

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