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Soliton-like Solutions of the Complex Non-linear Klein-Gordon Systems in $1+1$ Dimensions

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Abstract. In this paper, we present soliton-like solutions of the non-linear complex Klein-Gordon systems in 1+1 dimensions. We will use polar representation to introduce three different soliton-like solutions including, complex kinks (anti-kinks), radiative-profiles, and localized wave-packets. Complex kinks (anti-kinks) are topological objects with zero electrical charges. Radiative profiles are objects that move at the speed of light and therefore, have a zero rest mass. They can be created in kink-anti-kink collisions and vice versa. Localized wave packet solutions are non-topological objects for which wave and particle behavior are reconciled in a classical way. For localized wave packet solutions, the trivial initial phase imposes an uncertainty on the collision fates.

Keywords: complex, non-linear, Klein-Gordon, soliton, uncertainty, kink, radiative-profile, wave-packet.

1 Introduction

Nonlinear real Klein-Gordon (KG) systems in $1+1$ dimensions with topological kink (anti-kink) solutions have been suited for decades. The most well-known system in this area is the integrable sine-Gordon (SG) system [1, 2, 3, 4]. There are many other systems with kink solutions but are not as famous as the SG system [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In spite of wide studies in the real non-linear Klein-Gordon systems with soliton-like kink (anti-kink) solutions, the complex versions were to some extent out of interest [15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. An attempt in this line was the introduction of the complex kink (anti-kink) and radiative profiles, specially for the complex SG system [24].

In general, complex KG systems (Lagrangian densities) are ones which composed of complex fields. In other words, they are just special versions of the KG systems which are functional of scalar fields (ϕ) and their complex conjugates (ϕ^*). For complex KG systems, it is easy to show that the conservation of electrical charge is satisfied generally. In fact, it essentially originates from the $U(1)$ symmetry of such systems. For a real KG system with real scalar fields φ and standard kinetic terms $\partial_\mu\varphi\partial^\mu\varphi$, one can easily change it to a complex version via the following transformations: $\varphi \longrightarrow |\phi| = R = \sqrt{\phi\phi^*}$ and $\partial_\mu\varphi\partial^\mu\varphi \longrightarrow \partial_\mu\phi\partial^\mu\phi^*$.

In this paper, with a straightforward mathematical calculation, all soliton-like solutions of the complex non-linear Klein-Gordon systems (CNKG) in $1+1$ dimensions will be studied generally. Complex kinks (anti-kinks), radiative profiles and localized wave-packets are three types of soliton-like solutions which will be considered in details for all CNKG systems. For such systems, two kinds of conserved charge, topological and electrical, can be defined. All complex kink (anti-kink) solutions have the same rest mass and zero electrical charge but non-zero topological charge. Radiative profiles, as is obvious from their name, travel with the speed of light and have zero rest mass. Regarding radiative profiles, there is a countless variety. They can be topological or non-topological, have zero electric charge or non-zero one. Radiative profiles are created in collision between kinks and anti-kinks, and this process could be happening in the opposite direction; that means in the collision between two radiative profiles, kink-anti-kink pairs can be created.

Localized wave packet solutions, unlike the ordinary (linear) KG equations, do not disperse and they do satisfy a relation similar to the de Broglies wavelength-momentum relation. Two apparently contradictory aspects of quantum behavior, i.e. wave and particle behavior, are reconciled in a classical way for such soliton-like solutions. There are a continuous range of localized wave packet solutions which can be identified by different rest frequencies ω_o. Some of them are not stable and they decompose into a pair of separate kink and anti-kink. It will be shown that there is an uncertainty in the collision processes which is related to trivial initial phases. For different initial phases, particle aspect of the localized wave packets solutions remains unchanged, while the final behaviour in collision processes may be drastically affected. All soliton-like solutions were shown to obey the famous energy-rest mass-momentum relation of the special relativity.

We expect all CNKG systems to have similar features. Therefore, the complex ϕ^4 system as a special example of the CNKG systems in 1+1 dimensions will be employed. Fortunately, we will find well-known analytical functions for its complex kink (anti-kink) and localized wave-packet solutions. All numerical results in this paper will be prepared just for the complex ϕ^4 system. In fact, this paper is the complementary of the pervious paper [24] which was specially about radiative profiles and complex kinks (anti-kinks) of the complex SG system.

The organization of this paper is as follows: In the next section, we will introduce basic equations for non-linear complex KG systems in two different but equivalent representations. Sections 3, 4 and 5 contain a full discussion about complex kinks (anti-kink), radiative profiles and localized wave-packet solutions respectively. Section 6, contains a numerical study about complex kink-anti-kink collisions. In section 7, a numerical study will be prepare for the stability and uncertainty in collisions for the wave packet solutions. The last section devoted to the summary and conclusions.

2 Basic equations

In this paper, we use two different representations to introduce complex non-linear KG systems and related details in $1 + 1$ dimensions. Each of them can shed light on different aspect of the systems, but they are equivalent.

2.1 formal representation

Based on what is done in Refs. [1, 15, 16, 17, 18, 19, 20, 21, 22, 23], the non-linear complex Klein-Gordon systems can be generally introduced by the Lagrangian density as

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|), \tag{1}$$

where ϕ is a complex scalar field and $V(R)$ represents a self-interacting potential that depends only on the magnitude or module of ϕ ($R = |\phi|$). Using the least action principle, the dynamical equation for the evolution of ϕ can be obtained as follows:

$$\Box\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial V}{\partial \phi^*} = -\frac{1}{2}V'(|\phi|)\frac{\phi}{|\phi|}. \tag{2}$$

The energy-momentum tensor corresponding to the Lagrangian density (1) can be calculated using the Noethers theorem:

$$T^{\mu\nu} = 2\partial^{\mu}\phi^{*}2\partial^{\nu}\phi - g^{\mu\nu}\mathcal{L}, \tag{3}$$

where $g^{\mu\nu}$ is the Minkowski metric tensor. Also, the related energy density has the following form:

$$T^{00} = \varepsilon(x,t) = \frac{1}{c^2}\dot{\phi}\dot{\phi}^{*} + \acute{\phi}\acute{\phi}^{*} + V(|\phi|) \tag{4}$$

where the primes and dots denote space and time derivatives respectively.

Simply, it can be shown that the conservation relation is valid for two four-vector currents; the electrical current

$$j^{\mu} = i\eta(\phi^{*}\partial^{\mu}\phi - \phi\partial^{\mu}\phi^{*}), \tag{5}$$

and topological current

$$J^{\mu} = C\epsilon^{\mu\nu}\partial_{\nu}\phi. \tag{6}$$

Here $\epsilon^{\mu\nu}$ is an anti-symmetric tensor, C and η are just constant numbers. Corresponding to each of these currents, it can be easily shown that

$$q = \int_{-\infty}^{+\infty} j^{0}dx = \int_{-\infty}^{+\infty} i\eta(\phi^{*}\dot{\phi} - \phi\dot{\phi}^{*})dx, \tag{7}$$

and

$$Q = \int_{-\infty}^{+\infty} J^{0}dx = C[\phi(+\infty) - \phi(-\infty)], \tag{8}$$

that we call them electrical and topological charges respectively, which are constants of motion.

2.2 polar representation

Following the line of Refs. [1, 15, 16, 17, 18, 19, 20, 21], we can change variables ϕ and ϕ^{*} to polar fields $R(x,t)$ and $\theta(x,t)$ as defined by

$$\phi(x,t) = R(x,t)\exp[i\theta(x,t)]. \tag{9}$$

In terms of polar fields, the Lagrangian-density and field equations transform respectively to

$$\mathcal{L} = (\partial^{\mu}R\partial_{\mu}R) + R^{2}(\partial^{\mu}\theta\partial_{\mu}\theta) - V(R), \tag{10}$$

and

$$\Box R - R(\partial^{\mu}\theta\partial_{\mu}\theta) = -\frac{1}{2}\frac{dV}{dR}, \tag{11}$$

$$\partial_{\mu}(R^{2}\partial^{\mu}\theta) = 2R(\partial_{\mu}R\partial^{\mu}\theta) + R^{2}(\partial^{\mu}\partial_{\mu}\theta) = 0. \tag{12}$$

The related Hamiltonian (energy) density is obtained via the Noether's theorem:

$$\varepsilon(x,t) = \frac{1}{c^2}\dot{R}^2 + \acute{R}^2 + R^2(\frac{1}{c^2}\dot{\theta}^2 + \acute{\theta}^2) + V(R). \tag{13}$$

The corresponding electrical current is

$$j^{\mu} = -2\eta(R^{2}\partial^{\mu}\theta), \tag{14}$$

which according to field equation (12) is the conserved current.

3 Complex Kink (Anti-kink) solutions

If the phase function $\theta(x,t)$ is a constant, then equation (12) will be satisfied automatically and equation (11) is reduced to

$$\Box R = -\frac{1}{2}\frac{dV}{dR}, \tag{15}$$

which is the same as real non-linear Klein-Gordon equation with well-known kink and anti-kink solutions. If $R = \varphi_o(x)$ is the static kink (anti-kink) solution of the equation (15), then a moving complex kink (anti-kink) solution with velocity v would be introduced in the following form:

$$\phi(x,t) = \varphi_v(x,t)e^{i\theta} = \varphi_o(\pm\gamma(x - vt - x_o))e^{i\theta}. \tag{16}$$

in which $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$, and x_o is the initial position of the center of the kink (anti-kink). Generally, to have kink and anti-kink solutions, field potential $V(R)$ must have at least two successive non-negative vacuum points. For complex kinks and anti-kinks solutions (16), in general, it can be shown that the electrical charge (7) is always zero and the topological charge (8) will take the following final form:

$$Q = Ce^{i\theta}[\varphi_v(+\infty) - \varphi_v(-\infty)], \tag{17}$$

Moreover, using the energy-momentum tensor (3), it can be shown simply that in general, these solutions satisfy the relativistic energy-momentum relations as we expect for a real particle:

$$E_v = \gamma E_o = \int_{-\infty}^{+\infty}[\frac{1}{c^2}\dot{R}^2 + \acute{R}^2 + V(R)]dx, \tag{18}$$

$$p = \int_{-\infty}^{+\infty} T^{01}dx = \int_{-\infty}^{+\infty}[\frac{1}{c^2}\dot{\phi}\acute{\phi}]dx = \gamma m_o v, \tag{19}$$

where $E_o = m_o c^2$ is the rest energy of kinks (anti-kinks).

A real KG system with kink (anti-kink) solutions is the famous φ^4 system which is identified by the following Lagrangian density:

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi - (\varphi^2 - 1)^2. \tag{20}$$

Its standard kink (anti-kink) solution is

$$\varphi(x,t) = \tanh(\pm\gamma(x - vt - x_o)), \tag{21}$$

which varies between -1 and 1 (i.e. the successive vacuum points which are zeros of the potential $U(\varphi) = (\varphi^2 - 1)^2$). There are many works which studied the collisions and internal structures of the real φ^4 system [1, 5, 6, 7, 8, 9]. In this paper, we use this famous real KG system (20) to build the complex version of that. Therefore, the complex ϕ^4 system, as an example of the CNKG systems, can be introduced as follows:

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - [(R-1)^2 - 1]^2. \tag{22}$$

Note that, the module function R must be always positive. Therefore, according to Eq. (15), to have a kink (anti-kink) solution, since a kink (anti-kink) solution varies from one vacuum point to another one, we must use a modified version of the potential (instead of $U(R) = (R^2 - 1)^2$) in such a way that $R = 0, 2 \geq 0$ being vacuum points, i.e.

$$V(R) = [(R-1)^2 - 1]^2. \tag{23}$$

For this complex field system (22), the corresponding complex kink and anti-kink solutions are

$$\phi(x,t) = [\tanh(\pm\gamma(x - vt - x_o)) + 1]\, e^{i\theta}. \tag{24}$$

The related topological charge, if we choose $C = 1$, is

$$Q = \pm 2e^{i\theta}, \tag{25}$$

in which $+$ $(-)$ is for kinks (anti-kinks). In the next sections, specially as an example of the CNKG systems, we use the complex ϕ^4 system to introduce the other soliton-like solutions and study the fates of them in different collisions.

4 Radiative-profile solutions

If R_j is a vacuum point, i.e. $V(R_j) = 0$ and $\frac{dV}{dR}(R_j) = 0$, it is easy to see that equations (11) and (12) are satisfied for infinite soliton-like solutions which move at speed of light:

$$R(x,t) = R_j, \quad \theta(x,t) = f(x \pm ct), \tag{26}$$

where f is an arbitrary function. The related energy density (13) is now reduced to

$$\varepsilon(x,t) = 2R_j^2 \left(\frac{df}{d\widetilde{x}}\right)^2, \tag{27}$$

in which $\widetilde{x} = x \pm ct$. If for an arbitrary function $f(\widetilde{x})$, the corresponding energy density $\varepsilon(x,t)$ is localized, we have a "radiative-profile", i.e. a packet of energy which moves at the speed of light. Moreover, it can be proved generally that for such solutions (similar to massless particles), the relation between total energy and momentum is given by

$$p = \frac{\pm 1}{c}\int_{-\infty}^{+\infty} 2R_i^2 \left(\frac{df}{d\widetilde{x}}\right)^2 = \pm\frac{E}{c}, \tag{28}$$

Equivalently, we can use formal representation to introduce radiative profiles:

$$\phi(x,t) = R_j e^{if(x\pm ct)} = \phi_r(x \pm ct) + i\phi_i(x \pm ct), \tag{29}$$

provided

$$R_j = \sqrt{\phi_r^2 + \phi_i^2}. \tag{30}$$

Namely, for ϕ^4 system (23), to have a radiative profile, if we consider the form of the real part of the field as

$$\phi_r(x,t) = Ae^{-(x\pm ct)^2}, \tag{31}$$

we should consider the imaginary part as the following form:

$$\phi_i(x,t) = \sqrt{4 - A^2 e^{-2(x\pm ct)^2}}, \tag{32}$$

provided that $|A| < 2$. It can be simply conclude that this arbitrary constructed solution (relations (31) and (32)) is a non-topological object with zero topological charge. In general, for any arbitrary solution, depending on functions ϕ_i and ϕ_r to be non-topological (topological) objects, the related topological charge is zero (non-zero). Moreover, for radiative profiles, depending on functions ϕ_r and ϕ_i to be even or odd, the related electrical charges would be zero.

5 wave-packet solutions

It is easy to check that some solutions in the following form:

$$R(x,t) = R(\gamma(x-vt)), \qquad \theta(x,t) = k_\mu x^\mu + \theta_o = \omega t - kx + \theta_o, \tag{33}$$

satisfy equations (11) and (12), provided

$$k = \frac{\omega v}{c^2}, \tag{34}$$

and θ_o (initial phase) is just a constant. In general, $k^\mu \equiv (\omega, k)$ is defined as a $1+1$ vector, then $\partial^\mu \theta \partial_\mu \theta = k_\mu k^\mu = \omega_o^2/c^2$ is a constant scalar. Related to different values of ω_o, there are different wave equations for R (11):

$$\Box R = -\frac{d^2\phi}{d\overline{x}^2} = -\frac{1}{2}\frac{dV}{dR} + \frac{\omega_o^2}{c^2}R, \tag{35}$$

where

$$\overline{x} = \gamma(x - vt). \tag{36}$$

If we multiply (35) by $\frac{dR}{d\overline{x}}$ and integrate, it yields to

$$\left(\frac{dR(\overline{x})}{d\overline{x}}\right)^2 + \frac{\omega_o^2}{c^2}R^2 = V(R) + C', \tag{37}$$

where C' is an integration constant. This constant is expected to vanish for a localized wave-packet. This equation can be easily solved for R, once the potential $V(R)$ is known:

$$\overline{x} - x_o = \pm \int_{R_o}^{R} \frac{dR}{\sqrt{V(R) - \frac{\omega_o^2}{c^2}R^2}}. \tag{38}$$

In general, related to different values of ω_o, there are different non-topological solutions for $R(\overline{x})$. Exactly like complex kink energy-momentum relations (18) and (19), there are the same relations for wave-packet solutions:

$$E_v = \gamma E_o = \gamma \int_{-\infty}^{+\infty} [\acute{R}^2 + R^2\frac{\omega_o^2}{c^2} + V(R)]dx, \tag{39}$$

$$p = \frac{1}{c^2}\int_{-\infty}^{+\infty} [\dot{\phi}^* \acute{\phi} + \dot{\phi}\acute{\phi}^*]dx = \gamma m_o v, \tag{40}$$

where $E_o = m_o c^2$ is the rest energy of the wave-packet solution and is a function of ω_o. Moreover, one can use equation (34) to obtain

$$\omega = \gamma \omega_o. \tag{41}$$

Therefore, equations (39) and (41) show that frequency and energy have the same behavior and we can relate them via introducing a Planck-like constant $\overline{h}$:

$$E = \overline{h}\omega. \tag{42}$$

It is easy to understand that $\overline{h}$ is a function of rest frequency ω_o and for different wave-packet solutions, there are different $\overline{h}$ constants. Similarity, it is possible to find a relation between relativistic momentum of a soliton solution and wave number k:

$$p = \overline{h}k. \tag{43}$$

This equation is very interesting since it resembles the deBroglie's relation. Note that all Eqs. (34)-(43) were introduced similarly in Ref [22].

If we consider ϕ^4 system (23), the integral (38) can be easily performed, yielding the following solutions for $0 < w_o < 4$:

$$R(\overline{x}) = \frac{(4 - \omega_o^2/c^2)}{2 + \omega_o \cosh(\sqrt{4 - \omega_o^2/c^2}\ \overline{x})}. \tag{44}$$

Accordingly, there are infinite localized soliton-like wave-packet solutions which can be identified with different rest frequencies (ω_o). There are the similar works about the localized wave-packet solutions in the Refs. [22, 23].

6 Kink-anti-kink collisions in complex ϕ^4 system

In general, by preparing the suitable initial condition, we can study kink-anti-kink collisions numerically. For example, when an in-phase kink-anti-kink pair collides, i.e. $\theta_1 = \theta_2$, it treats exactly like what happened for the known real ϕ^4 system. An out-of-phase kink-anti-kink pair, i.e. $\theta_1 \neq \theta_2$, leads to the formation of a pair of radiative profiles in a desired stable localized form after collision (see Figs. 1). If in a kink-anti-kink collision, the phase difference $(\theta_2 - \theta_1)$ is equal to $\frac{\pi}{2}$, they always annihilate each other into two radiative profiles (Fig. 2).

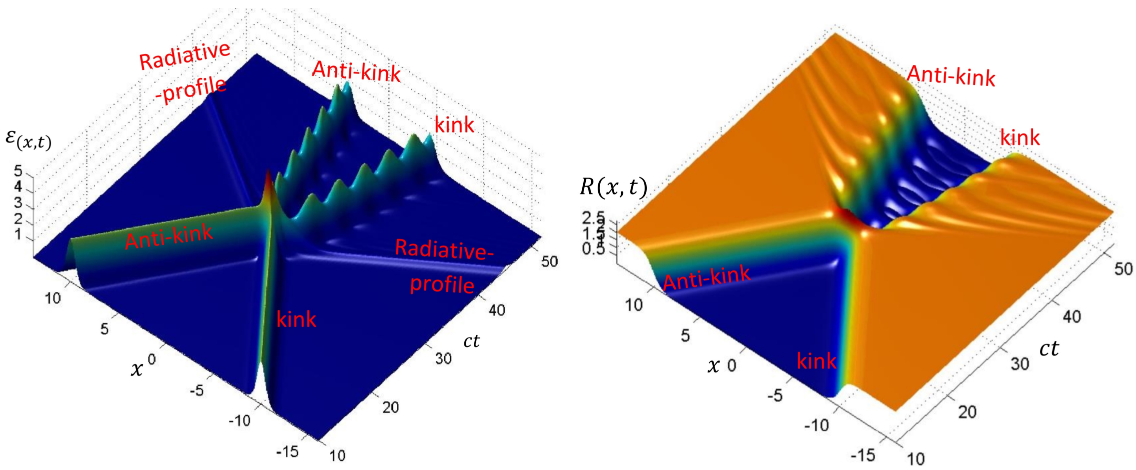

Figure 1: A complex kink ($\theta_1 = \arctan(\sqrt{2})$) collides with a complex anti-kink ($\theta_2 = \frac{\pi}{2}$) with initial kink (anti-kink) speed equals to $0.5c$. For a kink, contrary to an anti-kink, module function change from zero to 2. For radiative profiles, R is always constant ($R = 2$), hence it is not possible to track a radiative profile via the module representation.

Thus, we guess that the reversed process in collision between two radiative profiles would be possible. In other words, we expect we would be able to prepare a condition to create some kink-anti-kink pairs from the collisions of two radiative profiles. Theoretically, the discussion about this matter appears to be very difficult, but with the help of numerical simulations, we can see that in the case of the collision of two energetic radiative profiles, the creation of kink-anti-kink pairs with non-zero rest mass is possible. Although radiative profiles look like ordinary solutions of a linear wave equation, the inherent non-linearity in the original

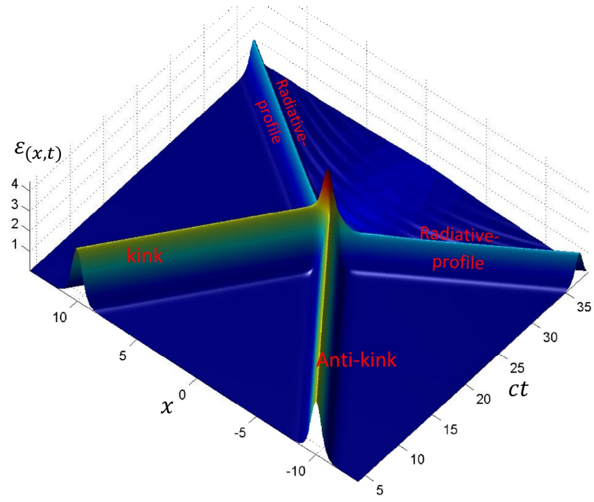

Figure 2: A complex kink (θ_1) collides with a complex anti-kink ($\theta_2 = \theta_1 + \frac{\pi}{2}$) with initial kink (anti-kink) speed of $0.5c$.

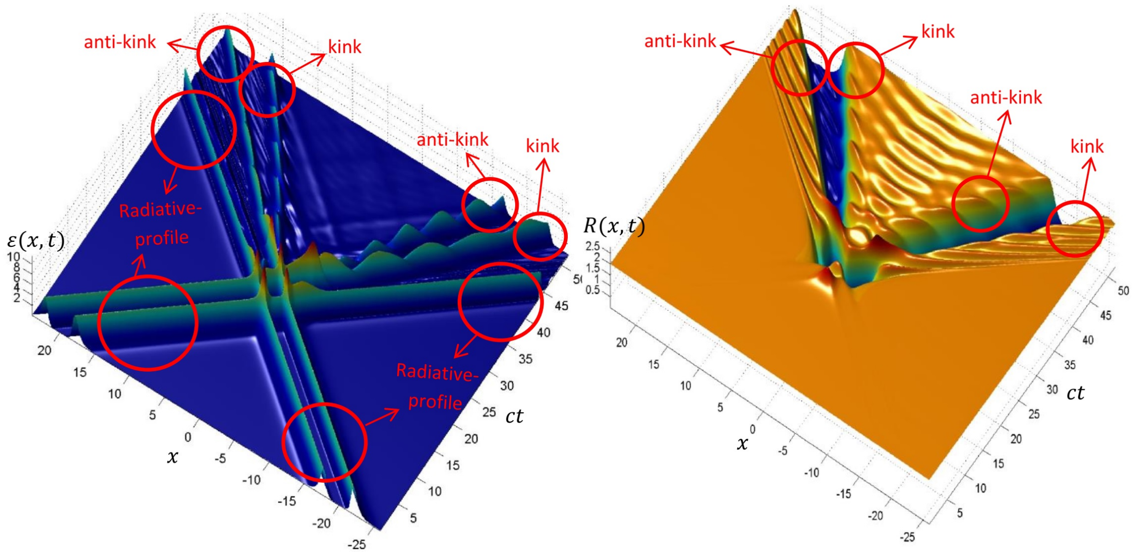

Figure 3: When two radiative profiles collide with each other, pairs of kink-anti-kinks can be created after collision.

field equation causes major differences. Namely, if a pair of similar radiative-profiles which are introduced by

$$\phi_r(x \pm ct) = \frac{2(x \pm ct)^6}{100 + (x \pm ct)^6}, \quad \phi_i = \sqrt{4 - \phi_r^2}, \tag{45}$$

collide with each other, two pairs of kink-anti-kink are created after collision (Fig. 3). It is completely evident that in all of these collisions electrical and topological charges are conserved. There are similar numerical results in the reference [24] which were obtained for the complex SG system.

7 The stability consideration and uncertainty in collisions for the localized wave packet solutions

As we said before, the total rest energy E_o for different localized wave packet solutions is a function of ω_o. For complex ϕ^4 system, the related curve of total rest energy versus ω_o^2/c^2 is shown in Fig. 4. The maximum of this curve occurs almost about $\omega_o^2/c^2 = 0.34$. Numerically, it was shown that for solutions which $\omega_o^2/c^2 < 0.34$, the wave packet solutions are completely unstable i.e. any small perturbation would change it to a pair of separate kink and anti-kink solutions (see Fig. 5). Moreover, for wave packet solutions which ω_o^2/c^2 is more close to 4, the more stability observed numerically.

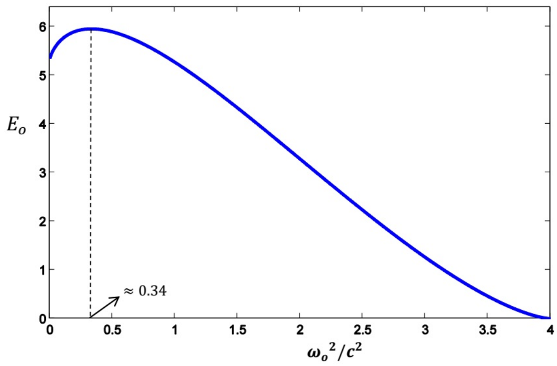

Figure 4: The total rest energy E_o versus ω_o^2/c^2 for different localized wave packet solutions of the complex ϕ^4 system.

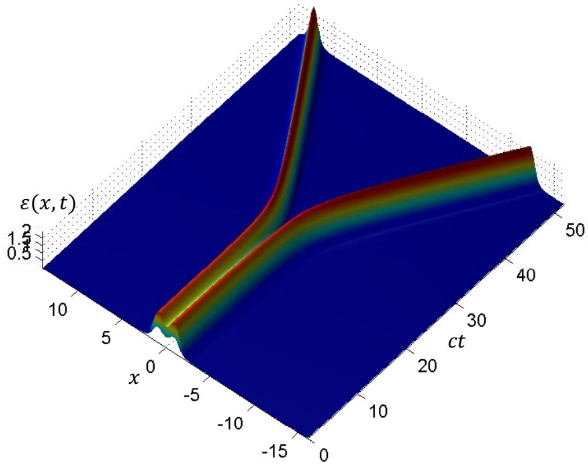

Figure 5: For a localized wave packet solution with $\omega_o^2/c^2 = 0.34$, any small perturbation changes it to a pair of separate kink and anti-kink.

One might think that optional initial phase θ_o for a localized wave packet solution is an unimportant parameter (33). In fact, it has no role in determining basic physical features of a single soliton such as energy, momentum and charge. But, it was seen numerically during the collision between localized wave packet solutions, these initial phases become very important. Namely, for two identical localized wave packet solutions with $\omega_o^2/c^2 = 0.7$ and initial speed $v = 0.5c$, if the initial phase difference is equal to π i.e. $\theta_{20} - \theta_{10} = \pi$, they are scattered from each other and reappear after collision. But if the initial phase difference is equal to 0, two pairs of kink-anti-kink would appear after collision (Fig. 6). Therefore, there is an apparent uncertainty in the collision processes which originates from the initial phases.

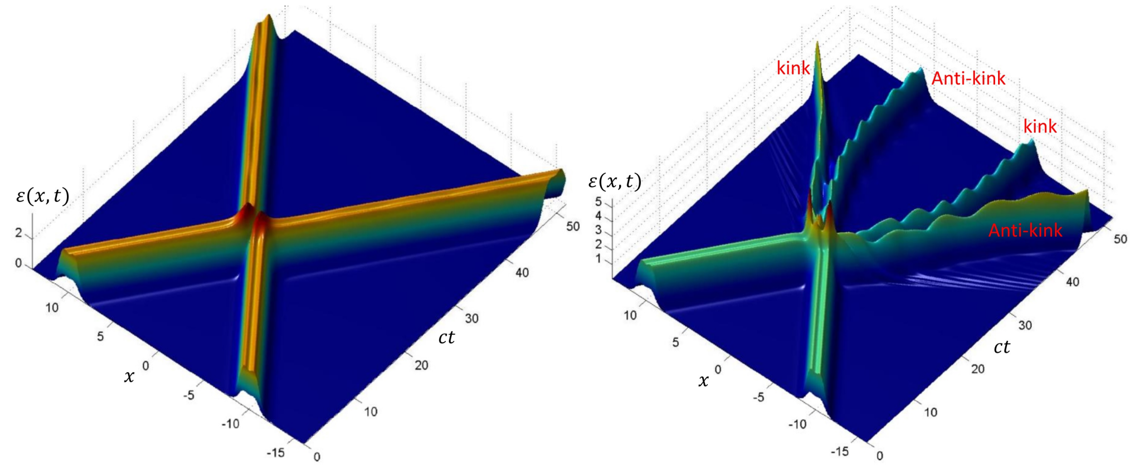

Figure 6: Two identical localized wave packet solutions with $\omega_o^2/c^2 = 0.7$ collide with each other. The initial speeds are $0.5c$. For the left (right) figure, the initial phase difference is π (0).

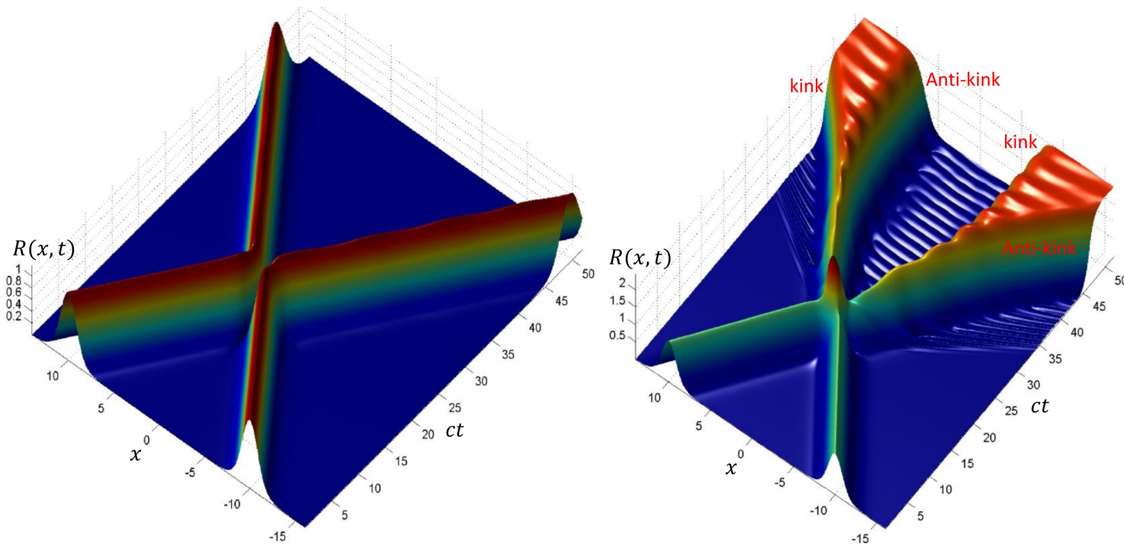

Figure 7: The module representation of Fig. 6. For a kink module function change from zero to 2 contrary to an anti-kink

8 Summery and conclusion

After reviewing some basic properties of the complex non-linear Klein-Gordon (CNKG) equations in two equivalent formal and polar representations, it was shown in general that for the CNKG equations in 1+1 dimensions, there are three different soliton-like solutions: complex kinks (anti-kinks), radiative profiles and localized wave-packets. Complex kinks (anti-kinks) are topological soliton-like solutions with zero electrical charge and the same rest mass. Radiative profiles are localized objects with zero rest mass and move at the speed of light. They can be topological or not, and the related charge may be zero or not. Localized wave-packet solutions are a Continuous range of the soliton-like solutions which can be identified by different rest frequencies (ω_o). Two apparently contradictory aspects of quantum behavior, i.e. wave and particle behavior, were reconciled in a classical way for such soliton-like solutions. All soliton-like solutions were shown to obey the famous energy-rest mass-momentum relation of the special relativity.

In the whole paper, complex ϕ^4 system as a special example of the non-linear complex KG systems in 1+1 dimensions was employed for better consideration of all of the soliton-like solutions. Kink-anti-kink collisions for the complex ϕ^4 system are studied numerically. It was seen that radiative-profiles always appear in out-of-phase kink-anti-kink collisions. In the reversed way, kink-anti-kink pairs can be created in collisions between radiative profiles. It was seen numerically that some of the localized wave packet solutions are completely unstable and decompose into a pair of separate kink-anti-kink solution. Moreover, we found numerically that there is an uncertainty in collision fates between localized wave packet solutions. This uncertainty originates from the initial phases. For different initial phases, particle aspect of the localized wave packets solutions remains unchanged, while the final behaviour may be drastically affected.

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