

## Structure formation and generalized second law of thermodynamics in some viable $f(R)$ -gravity models

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**Abstract.** We investigate the growth of matter density perturbations as well as the generalized second law (GSL) of thermodynamics in the framework of  $f(R)$ -gravity. We consider a spatially flat FRW universe filled with the pressureless matter and radiation which is enclosed by the dynamical apparent horizon with the Hawking temperature. For some viable  $f(R)$  models containing the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB models, we first explore numerically the evolution of some cosmological parameters like the Hubble parameter, the Ricci scalar, the deceleration parameter, the density parameters and the equation of state parameters. Then, we examine the validity of GSL and obtain the growth factor of structure formation. We find that for the aforementioned models, the GSL is satisfied from the early times to the present epoch. But in the farther future, the GSL for all models is violated. Our numerical results also show that for all models, the growth factor for larger structures, like the  $\Lambda$ CDM model, fit the data very well.

*Keywords:* Modified theories of gravity, Dark energy

## 1 Introduction

The observed accelerated expansion of the universe, as evidenced by a host of cosmological data such as supernovae Ia (SNeIa) [1], cosmic microwave background (CMB) [2, 4], large scale structure (LSS) [5], came as a great surprise to cosmologists. The present accelerated phase of the universe expansion reveals new physics missing from our universe's picture, and it constitutes the fundamental key to understand the fate of the universe. There are two representative approaches to explain the current acceleration of the universe. One is to introduce "dark energy" (DE) [8] in the framework of general relativity (GR). The other is to consider a theory of modified gravity (MG), such as  $f(R)$  gravity, in which the Einstein-Hilbert action in GR is generalized from the Ricci scalar  $R$  to an arbitrary function of the Ricci scalar [12]. Here, we will focus on the later approach. In [15], it was shown that a  $f(R)$  model with negative and positive powers of Ricci curvature scalar  $R$  can naturally combine the inflation at early times and the cosmic acceleration at late times. It is actually possible for viable  $f(R)$  models for late time acceleration to include inflation by adding  $R^2$  term. Therefore, it is natural to consider combined  $f(R)$  models which describe both primordial and present DE using one  $f(R)$  function, albeit one containing two greatly different characteristic energy scales [16, 17]. In [19], it was pointed out that the  $f(R)$ -gravity can also serve as dark matter (DM). In [20, 23], a set of  $f(R)$ -gravity models corresponding to different DE models were reconstructed. Although a great variety of  $f(R)$  models have been proposed in the literature, most of them is not perfect enough. A viable  $f(R)$  model should simultaneously satisfy stringent solar-system bounds on deviations from GR as well as accelerate the expansion at late times.

In order to distinguish between DE and MG, it is crucial to measure the growth of structure in addition to the expansion history. This is because any given expansion history predicted by a MG model could be emulated by a smooth DE component. Measuring the matter velocity field at the locations of the galaxies via spectroscopy helps differentiate between the effect of DE and MG as the source of the accelerating universe through measurements of Redshift Space Distortions (RSD) [25]. RSD was identified by the recent ‘‘Rocky III’’ report as among the most powerful ways of addressing whether the acceleration is caused by DE or MG [26]. For the case of DE model, the growth index is independent of the size of structure, as the structure formation equation for the scales larger than the Jeans length is independent of the wavenumbers while in the MG model, the effective gravitational constant relates the growth index of the structure to its size [27, 28, 29, 30]. An interesting feature of the  $f(R)$  theories is the fact that the gravitational constant in  $f(R)$ -gravity, varies with length scale as well as with time. Thus, the evolution of the matter density perturbation,  $\delta_m \equiv \delta\rho_m/\rho_m$ , in this theory is affected by the effective Newton coupling constant,  $G_{\text{eff}}$ , and it is scale dependent, too. Therefore, the matter density perturbation is a crucial tool to distinguish MG from DE model in GR, in particular the standard  $\Lambda$ CDM model. The scale dependencies of the linear growth rate of metric and density perturbations in  $f(R)$ -gravity can change predictions for cosmological power spectra in the linear regime [31].

On the other hand, the connection between gravity and thermodynamics is one of surprising features of gravity which was first reinforced by Jacobson [32], who associated the Einstein field equations with the Clausius relation in the context of black hole thermodynamics. This idea was also extended to the cosmological context and it was shown that the Friedmann equations in the Einstein gravity [33] can be written in the form of the first law of thermodynamics (the Clausius relation). The equivalence between the first law of thermodynamics and the Friedmann equation was also found for  $f(R)$ -gravity [34]. Besides the first law, the generalized second law (GSL) of gravitational thermodynamics, which states that entropy of the fluid inside the horizon plus the geometric entropy do not decrease with time, was also investigated in  $f(R)$ -gravity [36]. The GSL of thermodynamics in the accelerating universe driven by DE or MG has been also studied extensively in the literature [37]-[58].

All these motivate us to investigate the growth of matter density perturbations in a class of metric  $f(R)$  models and see scale dependence of growth factor. Additionally, we are interested in examining the validity of GSL in some viable  $f(R)$ -gravity models. The structure of this paper is as follows. In Sec. 2, within the framework of  $f(R)$ -gravity we consider a spatially flat Friedmann-Robertson-Walker (FRW) universe filled with the pressureless matter and radiation. In Sec. 3, we study the growth rate of matter density perturbations in  $f(R)$ -gravity. In Sec. 4, the GSL of thermodynamics on the dynamical apparent horizon with the Hawking temperature is explained. In Sec. 5, the cosmological evolution of  $f(R)$  models is illustrated. In Sec. 6, the viability conditions for  $f(R)$  models are discussed. In addition, some viable  $f(R)$  models containing the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB models are introduced. In Sec. 7, we give numerical results obtained for the evolution of some cosmological parameters, the GSL and the growth of structure formation in the aforementioned  $f(R)$  models. Finally, Sec. 8 is devoted to conclusions.

## 2 $f(R)$ -gravity framework

Within the framework of  $f(R)$ -gravity, the modified Einstein-Hilbert action in the Jordan frame is given by [12]

$$S_J = \int \sqrt{-g} d^4x \left[ \frac{f(R)}{16\pi G} + L_{\text{matter}} \right], \quad (1)$$

where  $G$ ,  $g$ ,  $R$  and  $L_{\text{matter}}$  are the gravitational constant, the determinant of the metric  $g_{\mu\nu}$ , the Ricci scalar and the lagrangian density of the matter inside the universe, respectively. Also,  $f(R)$  is an arbitrary function of the Ricci scalar.

Varying the action (1) with respect to  $g_{\mu\nu}$  yields

$$FG_{\mu\nu} = 8\pi GT_{\mu\nu}^{(m)} - \frac{1}{2}g_{\mu\nu}(RF - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu}\square F. \quad (2)$$

Here,  $F = df/dR$ ,  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$  and  $T_{\mu\nu}^{(m)}$  is the energy-momentum tensor of the matter. The gravitational field equations (2) can be rewritten in the standard form as [59, 61]

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(D)}), \quad (3)$$

with

$$8\pi GT_{\mu\nu}^{(D)} = (1 - F)G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(RF - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu}\square F. \quad (4)$$

For a spatially flat FRW metric, taking  $T_{\nu}^{\mu(m)} = \text{diag}(-\rho, p, p, p)$  in the perfect fluid form, then the set of field equations (3) reduce to the modified Friedmann equations in the framework of  $f(R)$ -gravity as [62]

$$3H^2 = 8\pi G(\rho + \rho_D), \quad (5)$$

$$2\dot{H} = -8\pi G(\rho + \rho_D + p + p_D), \quad (6)$$

where

$$8\pi G\rho_D = \frac{1}{2}(RF - f) - 3H\dot{F} + 3H^2(1 - F), \quad (7)$$

$$8\pi Gp_D = \left[ \frac{-1}{2}(RF - f) + \ddot{F} + 2H\dot{F} - (1 - F)(2\dot{H} + 3H^2) \right], \quad (8)$$

with

$$R = 6(\dot{H} + 2H^2). \quad (9)$$

Here,  $H = \dot{a}/a$  is the Hubble parameter. Also,  $\rho_D$  and  $p_D$  are the curvature contribution to the energy density and pressure which can play the role of DE. Also,  $\rho = \rho_{\text{BM}} + \rho_{\text{DM}} + \rho_{\text{rad}}$  and  $p = p_{\text{rad}} = \rho_{\text{rad}}/3$  are the energy density and pressure of the matter inside the universe, consist of the pressureless baryonic and dark matters as well as the radiation. On the whole of the paper, the dot and the subscript  $R$  denote the derivatives with respect to the cosmic time  $t$  and the Ricci scalar  $R$ , respectively.

The energy conservation laws are still given by

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (10)$$

$$\dot{\rho}_{\text{rad}} + 4H\rho_{\text{rad}} = 0, \quad (11)$$

$$\dot{\rho}_D + 3H(\rho_D + p_D) = 0, \quad (12)$$

where  $\rho_m = \rho_{\text{BM}} + \rho_{\text{DM}}$ . From Eqs. (10) and (11) one can find

$$\rho = \frac{\rho_{m_0}}{a^3} + \frac{\rho_{\text{rad}_0}}{a^4}, \quad (13)$$

where  $\rho_{m_0} = \rho_{\text{BM}_0} + \rho_{\text{DM}_0}$  and  $\rho_{\text{rad}_0}$  are the present values of the energy densities of matter and radiation. We also choose  $a_0 = 1$  for the recent value of the scale factor.

Using the usual definitions of the density parameters

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{8\pi G \rho_{m_0}}{3H^2 a^3}, \quad \Omega_{\text{rad}} = \frac{\rho_{\text{rad}}}{\rho_c} = \frac{8\pi G \rho_{\text{rad}_0}}{3H^2 a^4}, \quad \Omega_D = \frac{\rho_D}{\rho_c} = \frac{8\pi G \rho_D}{3H^2}, \quad (14)$$

in which  $\rho_c = 3H^2/(8\pi G)$  is the critical energy density, the modified Friedmann equation (5) takes the form

$$1 = \Omega_m + \Omega_{\text{rad}} + \Omega_D. \quad (15)$$

From the energy conservation (12), the equation of state (EoS) parameter due to the curvature contribution is defined as

$$\omega_D = \frac{p_D}{\rho_D} = -1 - \frac{\dot{\rho}_D}{3H\rho_D}. \quad (16)$$

Using the modified Friedmann equations (5) and (6), the effective EoS parameter is obtained as

$$\omega_{\text{eff}} = \frac{p + p_D}{\rho + \rho_D} = -1 - \frac{2\dot{H}}{3H^2}. \quad (17)$$

Also, the two important observational cosmographic parameters called the deceleration  $q$  and the jerk  $j$  parameters, respectively related to  $\ddot{a}$  and  $\dddot{a}$ , are given by [64]

$$q = -\frac{\ddot{a}}{aH^2} = -1 - \frac{\dot{H}}{H^2} = 1 - \frac{R}{6H^2}, \quad (18)$$

$$j = \frac{\dddot{a}}{aH^3} = 1 - \frac{\dot{H}}{H^2} + \frac{\dot{R}}{6H^3} = 2 + q + \frac{\dot{R}}{6H^3}. \quad (19)$$

Cosmologists believe that the universe transitioned from deceleration to acceleration in a cosmic jerk. The deceleration to acceleration transition of the universe occurs for different models with a positive value of the jerk parameter and negative value of the deceleration parameter [67]. For example, flat  $\Lambda$ CDM models have a constant jerk  $j = 1$  [70].

### 3 Growth rate of matter density perturbations

Here, we study the growth rate of matter density perturbations in  $f(R)$ -gravity. The origin of structure formation in the universe is seeded by the small quantum fluctuations generated at the inflationary epoch. These small perturbations over time grew to become all of the structure we observe. Once the universe becomes matter dominated, primeval density inhomogeneities ( $\delta\rho_m/\rho_m \sim 10^{-5}$ ) are amplified by gravity and grow into the structure we see today [71].

The evolution of the matter density contrast  $\delta_m = \delta\rho_m/\rho_m$  provides an important tool to distinguish  $f(R)$ -gravity and generally MG models from DE inside GR and, in particular, from the  $\Lambda$ CDM model. We consider the linear scalar perturbations around a flat FRW background in the Newtonian (longitudinal) gauge as

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)dx^2, \quad (20)$$

with two scalar potentials  $\Psi$  and  $\Phi$  describing the perturbations in the metric. In this gauge, the matter density perturbation  $\delta_m$  and the perturbation of  $\delta F(R)$  obey the following equations in the Fourier space [72, 73]

$$\ddot{\delta}_m + \left(2H + \frac{\dot{F}}{2F}\right) \dot{\delta}_m - \frac{8\pi G \rho_m}{2F} \delta_m = \frac{1}{2F} \left[ \left(-6H^2 + \frac{k^2}{a^2}\right) \delta F + 3H\dot{\delta F} + 3\ddot{\delta F} \right], \quad (21)$$

$$\delta\ddot{F} + 3H\dot{\delta F} + \left(\frac{k^2}{a^2} + \frac{F}{3F_R} - \frac{R}{3}\right) \delta F = \frac{8\pi G}{3} \rho_m \delta_m + \dot{F} \dot{\delta}_m, \quad (22)$$

where  $k$  is the comoving wave number. For the modes deep inside the Hubble radius ( $k^2/a^2 \gg H^2$ ), with considering this fact that the time derivative of  $F$  is small ( $|\dot{F}| \ll HF$ ) and with neglecting the oscillating mode of  $\delta F$  ( $\delta\ddot{F} \ll H\dot{\delta F} \ll H^2$ ), the evolution of matter density contrast  $\delta_m$  is govern by [74, 75, 77]

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}} \rho_m \delta_m = 0, \quad (23)$$

where

$$G_{\text{eff}} = \frac{G}{F} \left[ \frac{4}{3} - \frac{1}{3} \frac{M^2 a^2}{k^2 + M^2 a^2} \right], \quad (24)$$

and  $M^2 = \frac{F}{3F_R}$ . The fraction of effective gravitational constant to the Newtonian one, i.e.  $G_{\text{eff}}/G$ , is defined as screened mass function in the literature [29]. Equation (24) obviously shows that the screened mass function is the time and scale dependent parameter.

With the help of new variable namely  $g(a) = \delta_m/a$  which parameterizes the growth of structure in the matter, Eq. (23) becomes

$$\frac{d^2 g}{d \ln a^2} + \left(4 + \frac{\dot{H}}{H^2}\right) \frac{dg}{d \ln a} + \left(3 + \frac{\dot{H}}{H^2} - \frac{4\pi G_{\text{eff}} \rho_m}{H^2}\right) g = 0. \quad (25)$$

In general, there is no analytical solution to this equation. But in [79] for an asymptotic form of viable  $f(R)$  models at high curvature regime given by  $f(R) = R + R^{-n}$  where  $n > -1$ , an analytic solution for density perturbations in the matter component during the matter dominated stage was obtained in terms of hypergeometric functions. In what follows, we solve the differential equation (25), numerically. To this aim, the natural choice for the initial conditions are  $g(a_m) = 1$  and  $\frac{dg}{d \ln a} |_{a=a_m} = 0$ , where  $a_m = 1/(1+z_m)$  should be taken during the matter era, because for the matter dominated universe, i.e.  $H^2 = 8\pi G \rho_m/3$  and  $G_{\text{eff}}/G = 1$ , the solution of Eq. (23) yields  $\delta_m = a$ . The other useful quantity is the logarithmic rate of change of matter density with respect to the scale factor, known as the growth factor. The growth factor is defined as [80]

$$f(z) = \frac{d \ln \delta_m}{d \ln a} = -(1+z) \frac{d \ln \delta_m}{dz}, \quad (26)$$

which is an observational parameter. The redshift space distortion is used as a probe to measure the growth rate of the structures,  $f(z)$ , to underpin the expansion history of the universe and to distinguish between MG and DE theories [30]. In the present work, we obtain the evolution of linear perturbations relevant to the matter spectrum for the scales;  $k = 0.1, 0.01, 0.001 h \text{ Mpc}^{-1}$ , where  $h$  corresponds to the Hubble parameter today. For smaller scales,  $k > 0.2 h \text{ Mpc}^{-1}$ , the effect of non-linearity becomes important. In the non-linear regime, while gravity is still in the weak field limit, density fluctuations are no longer small and in addition, the density or potential fields may couple to additional scalar fields introduced in MG theories. The non-linear regime is therefore the hardest to describe in any general way as the nature of the coupling to scalar fields is theory specific [82].

## 4 Generalized second law of thermodynamics

Here, we are interested in examining the validity of the GSL of gravitational thermodynamics for a given  $f(R)$  model. According to the GSL, entropy of the matter inside the horizon beside the entropy associated with the surface of horizon should not decrease during the time [33]. As demonstrated by Bekenstein, this law is satisfied by black holes in contact with their radiation [83]. The entropy of the matter containing the pressureless matter and radiation inside the horizon is given by the Gibbs' equation [37]

$$T_A dS = dE + p dV, \quad (27)$$

where  $E = (\rho_m + \rho_{\text{rad}})V$ ,  $V = \frac{4\pi}{3}\tilde{r}_A^3$  is the volume containing the matter with the radius of the dynamical apparent horizon  $\tilde{r}_A = (H^2 + \frac{K}{a^2})^{-1/2}$  and  $T_A = \frac{1}{2\pi\tilde{r}_A} (1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})$  is the Hawking temperature. Here,  $p = p_{\text{rad}} = \rho_{\text{rad}}/3$  is the total pressure of the matter inside the universe, consist of the pressureless baryonic and dark matters as well as the radiation. Taking time derivative of Eq. (27) and using the energy equations (10)-(11) as well as the Friedmann equations (5)-(6) for a spatially flat FRW universe ( $K = 0$ ), one can find

$$T_A \dot{S} = \frac{\tilde{r}_A^2}{2G} (\dot{\tilde{r}}_A - H\tilde{r}_A) \left( -2\dot{H} + H \frac{d}{dt} - \frac{d^2}{dt^2} \right) F. \quad (28)$$

The horizon entropy in  $f(R)$ -gravity is given by  $S_A = \frac{AF}{4G}$  [84], where  $A = 4\pi\tilde{r}_A^2$  is the area of the apparent horizon. Taking the time derivative of  $S_A$ , one can get the evolution of horizon entropy as

$$T_A \dot{S}_A = \frac{1}{4GH} (2H\tilde{r}_A - \dot{\tilde{r}}_A) \left( \frac{2\dot{\tilde{r}}_A}{\tilde{r}_A} + \frac{d}{dt} \right) F. \quad (29)$$

Now, we can calculate the GSL due to different contributions of the matter and horizon. Adding Eqs. (28) and (29), one can get the GSL in  $f(R)$ -gravity as [36]

$$T_A \dot{S}_{\text{tot}} = \frac{1}{4GH^4} \left[ 2\dot{H}^2 F - \dot{H}H\dot{F} + 2(\dot{H} + H^2)\ddot{F} \right], \quad (30)$$

where  $S_{\text{tot}} = S + S_A$ . Note that Eq. (30) shows that the validity of the GSL, i.e.  $T_A \dot{S}_{\text{tot}} \geq 0$  depends on the  $f(R)$ -gravity model. For the Einstein gravity ( $F = 1$ ), one can immediately find that the GSL (30) reduces to

$$T_A \dot{S}_{\text{tot}} = \frac{\dot{H}^2}{2GH^4} \geq 0, \quad (31)$$

which shows that the GSL is always fulfilled throughout history of the universe. Within the framework of Einstein's gravity, it was also shown that the GSL in the presence of DE is always satisfied during history of the universe [37]. The GSL of thermodynamics is a universal principle governing the universe. As is well known, the GSL is a powerful tool to set bounds on astrophysical and cosmological models [86]. The satisfaction of the GSL of thermodynamics provides further confidence on the thermodynamical interpretation of gravity in  $f(R)$  scenario based on the profound connection between gravity and thermodynamics. Therefore, as one of the most important theoretical touchstones to examine whether  $f(R)$ -gravity can be an alternative gravitational theory to GR, we examine the validity of the GSL for some viable  $f(R)$  models in subsequent sections.

## 5 Cosmological evolution

Here, we recast the differential equations governing the evolution of the universe in dimensionless form which is more suitable for numerical integration. To do so, following [91] we use the dimensionless quantities

$$\bar{t} = H_0 t, \quad \bar{H} = \frac{H}{H_0}, \quad \bar{R} = \frac{R}{H_0^2}, \quad (32)$$

$$\bar{f} = \frac{f}{H_0^2}, \quad \bar{F} = F, \quad \bar{F}_R = \frac{F_R}{H_0^{-2}}, \quad \bar{F}_{RR} = \frac{F_{RR}}{H_0^{-4}}, \quad (33)$$

where  $H_0$  is the Hubble parameter today. With the help of the above definitions and using

$$\frac{d}{d\bar{t}} = -\bar{H}(1+z) \frac{d}{dz}, \quad (34)$$

one can rewrite the modified Friedmann equation (5) as follows

$$\bar{H}^2 = \Omega_{\text{m}_0} [(1+z)^3 + \chi(1+z)^4] + (\bar{F}-1) [\bar{H}^2 - (1+z)\bar{H}\bar{H}'] - \frac{1}{6}(\bar{f}-\bar{R}) + (1+z)\bar{H}^2\bar{F}_R\bar{R}', \quad (35)$$

where  $\chi = \rho_{\text{rad}_0}/\rho_{\text{m}_0} = \Omega_{\text{rad}_0}/\Omega_{\text{m}_0}$  and prime “'” denotes a derivative with respect to the cosmological redshift  $z = \frac{1}{a} - 1$ .

To solve Eq. (35), we introduce new variables as [92]:

$$y_{\text{H}} := \frac{\rho_{\text{D}}}{\rho_{\text{m}_0}} = \frac{\bar{H}^2}{\Omega_{\text{m}_0}} - (1+z)^3 - \chi(1+z)^4, \quad (36)$$

and

$$y_{\text{R}} := \frac{\bar{R}}{\Omega_{\text{m}_0}} - 3(1+z)^3. \quad (37)$$

Taking the derivative of both sides of Eqs. (36) and (37) with respect to redshift  $z$  yield

$$-(1+z)y_{\text{H}}' = \frac{1}{3}y_{\text{R}} - 4y_{\text{H}}, \quad (38)$$

$$\begin{aligned} -(1+z)y_{\text{R}}' &= 9(1+z)^3 - \frac{1}{\bar{H}^2\bar{F}_R} \left\{ y_{\text{H}} + \frac{1}{6\Omega_{\text{m}_0}}(\bar{f}-\bar{R}) \right. \\ &\quad \left. - (\bar{F}-1) \left[ \frac{y_{\text{R}}}{6} - y_{\text{H}} - \frac{1}{2} \left( (1+z)^3 + 2\chi(1+z)^4 \right) \right] \right\}. \end{aligned} \quad (39)$$

Finally, inserting Eq. (39) into the derivative of Eq. (38) gives a second differential equation governing  $y_{\text{H}}(z)$  as [93]

$$(1+z)^2 y_{\text{H}}'' + J_1(1+z)y_{\text{H}}' + J_2 y_{\text{H}} + J_3 = 0, \quad (40)$$

where

$$J_1 = -3 - \left( \frac{1-\bar{F}}{6\bar{H}^2\bar{F}_R} \right), \quad (41)$$

$$J_2 = \frac{2-\bar{F}}{3\bar{H}^2\bar{F}_R}, \quad (42)$$

$$J_3 = -3(1+z)^3 - \frac{1}{6\bar{H}^2\bar{F}_R} \left[ (1-\bar{F}) \left( (1+z)^3 + 2\chi(1+z)^4 \right) + \frac{1}{3\Omega_{m_0}} (\bar{R} - \bar{f}) \right]. \quad (43)$$

Equation (40) cannot be solved analytically. Hence, we need to solve it numerically. To do so, we use the two initial conditions  $y_H(z_i) = 3$  and  $y'_H(z_i) = 0$  which come from the  $\Lambda$ CDM approximation of  $f(R)$  model in high curvature regime. Notice  $z_i$  is the proper redshift in which we have  $RF_R(z_i) \leq 10^{-13}$ .

With the help of Eqs. (14), (16), (17) and (36), one can obtain the evolutionary behaviors of the matter density parameter,  $\Omega_m(z)$ , DE density parameter,  $\Omega_D(z)$ , EoS parameter of DE,  $\omega_D(z)$ , and effective EoS parameter,  $\omega_{\text{eff}}(z)$ , in terms of  $y_H$  and its derivatives as follows

$$\Omega_m(z) = \frac{(1+z)^3}{y_H + (1+z)^3 + \chi(1+z)^4}, \quad (44)$$

$$\Omega_D(z) = \frac{y_H}{y_H + (1+z)^3 + \chi(1+z)^4}, \quad (45)$$

$$\omega_D(z) = -1 + \frac{1+z}{3} \left( \frac{y'_H}{y_H} \right), \quad (46)$$

$$\omega_{\text{eff}}(z) = -1 + \frac{(1+z)}{3} \left[ \frac{y'_H + 3(1+z)^2 + 4\chi(1+z)^3}{y_H + (1+z)^3 + \chi(1+z)^4} \right]. \quad (47)$$

Also from Eqs. (18), (19) and (36) one can get the evolutions of the deceleration and jerk parameters as

$$q(z) = -1 + \frac{(1+z)}{2} \left[ \frac{y'_H + 3(1+z)^2 + 4\chi(1+z)^3}{y_H + (1+z)^3 + \chi(1+z)^4} \right], \quad (48)$$

$$j(z) = 1 + \frac{(1+z)}{2} \left[ \frac{(1+z)y''_H - 2y'_H + 4\chi(1+z)^3}{y_H + (1+z)^3 + \chi(1+z)^4} \right]. \quad (49)$$

## 6 Viable $f(R)$ -gravity models

The necessary conditions for having a viable  $f(R)$  model can be summarized as follows:

(i)  $F > 0$ , which keeps the positivity of the effective gravitational coupling constant and avoids anti-gravity.

(ii)  $F_R > 0$ , which gives the stability condition of cosmological perturbations [31, 96, 97].

(iii) In the large curvature regime ( $R/R_0 \gg 1$ ), the  $f(R)$  model behaves like  $\Lambda$ CDM model. It means that  $f(R) \rightarrow R - 2\Lambda$ , where  $R_0$  is the Ricci scalar today. This is required for the presence of the matter-dominated stage.

(iv) A stable late time de Sitter point; the condition which is required for this stability is,  $0 < m(R = R_d) < 1$  [98], where  $m \equiv \frac{RF_R}{F}$  and  $R_d = 2f(R_d)/F(R_d)$  is the value of the scalar curvature at the de Sitter point. Note that the quantity  $m$  characterizes the deviation from the  $\Lambda$ CDM model.

(v) Passing constraint from the equivalence principle and solar system test [99].

Since we are interested in investigating the growth of structure formation and examining the GSL in  $f(R)$ -gravity, hence in what follows we consider some viable  $f(R)$  models including the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB models which satisfy the conditions (i) to (v).



## 6.1 Starobinsky Model

The Starobinsky  $f(R)$  model is as follows [74]

$$f(R) = R + \lambda R_s \left[ \left( 1 + \frac{R^2}{R_s^2} \right)^{-n} - 1 \right], \quad (50)$$

where  $n > 0$ ,  $\lambda$  and  $R_s$  are constant parameters of the model. Following [100], we take  $n = 2$  and  $\lambda = 1$ . Note that in the high  $z$  regime ( $z \simeq z_i$ ), we have  $R/R_s \gg 1$ . This yields the  $f(R)$  model (50) to behave like the  $\Lambda$ CDM model, i.e.  $f(R) = R - 2\Lambda$ . Consequently, the constant parameter  $R_s$  is obtained as  $R_s = 18\Omega_{m_0}H_0^2/\lambda$ .

## 6.2 Hu-Sawicki Model

This model was reconstructed based on the local observational data and presented by Hu and Sawicki [92] as

$$f(R) = R - \frac{c_1 R_s \left(\frac{R}{R_s}\right)^n}{c_2 \left(\frac{R}{R_s}\right)^n + 1}, \quad (51)$$

where  $n > 0$ ,  $c_1, c_2$  and  $R_s$  are constants of the model. For this model, we take  $n = 4$ ,  $c_1 = 1.25 \times 10^{-3}$ ,  $c_2 = 6.56 \times 10^{-5}$  [91], and obtain  $R_s = 18c_2\Omega_{m_0}H_0^2/c_1$ .

## 6.3 Exponential Model

This model is defined by the following function [93],

$$f(R) = R - \beta R_s \left( 1 - e^{-\frac{R}{R_s}} \right), \quad (52)$$

where  $\beta$  and  $R_s$  are two constants of the model. Here,  $R_s$  corresponds to the characteristic curvature modification scale. Here, we take  $\beta = 1.8$  [93] and obtain  $R_s = 18\Omega_{m_0}H_0^2/\beta$ .

## 6.4 Tsujikawa Model

This model was originally presented in [73] as

$$f(R) = R - \lambda R_s \tanh \left( \frac{R}{R_s} \right), \quad (53)$$

where  $\lambda$  and  $R_s$  are the model parameters. For this model, we obtain  $R_s = 18\Omega_{m_0}H_0^2/\lambda$  and set  $\lambda = 1$  [101].

## 6.5 AB Model

This model was proposed by Appleby and Battye [16, 103] as

$$f(R) = \frac{R}{2} + \frac{\epsilon}{2} \log \left[ \frac{\cosh \left( \frac{R}{\epsilon} - b \right)}{\cosh(b)} \right], \quad (54)$$

where  $b$  is a dimensionless constant and  $\epsilon = R_s/[b + \log(2 \cosh b)]$ . The constant  $R_s$  can be obtained at high curvature regime when the AB  $f(R)$  model (54) behaves like the  $\Lambda$ CDM model, i.e.  $f(R) = R - 2\Lambda$ . This gives

$$R_s = \frac{-36 \Omega_{m_0} H_0^2 [b + \log(2 \cosh b)]}{\log \left( \frac{1 - \tanh b}{2} \right)}.$$

Here, we also set  $b = 1.4$ .

## 7 Numerical results

Here to solve Eq. (40) numerically, we choose the cosmological parameters  $\Omega_{m_0} = 0.24$ ,  $\Omega_{D_0} = 0.76$  and  $\Omega_{\text{rad}_0} = 4.1 \times 10^{-5}$ . As we have already mentioned, we use the two suitable initial conditions  $y_{\text{H}}(z_i) = 3$  and  $y'_{\text{H}}(z_i) = 0$ , in which  $z_i$  is obtained where  $RF_{\text{R}} \rightarrow 10^{-13}$ . For the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB  $f(R)$  models, we obtain  $z_i = 15.61, 13.12, 3.66, 3.52$  and  $3.00$ , respectively.

In addition, to study the growth rate of matter density perturbations, we numerically solve Eq. (25) with the initial conditions  $g(z_m) = 1$  and  $(dg/d \ln a)|_{z_m} = 0$ , in which  $z_m$  is obtained where  $\Omega_m(z_m) = 1$ . For the aforementioned models, we obtain  $z_m = 14, 13, 12, 14$  and  $14.36$ , respectively.

With the help of numerical results obtained for  $y_{\text{H}}(z)$  in Eq. (40), we can obtain the evolutionary behaviors of  $H, R, q, \Omega_m, \Omega_D, \omega_{\text{eff}}, \omega_D$  and GSL for our selected  $f(R)$  models. The results for the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB  $f(R)$  models are displayed in Figs. 1-5. Figures show that: (i) the Hubble parameter and the Ricci scalar decrease during history of the universe. (ii) The deceleration parameter  $q$  varies from an early matter-dominant epoch ( $q = 0.5$ ) to the de Sitter era ( $q = -1$ ) in the future, as expected. It also shows a transition from a cosmic deceleration  $q > 0$  to the acceleration  $q < 0$  in the near past. The current values of the deceleration parameter for the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB  $f(R)$  models are obtained as  $q_0 = -0.56, -0.60, -0.56, -0.57$  and  $-0.60$ , respectively. These are in good agreement with the recent observational constraint  $q_0 = -0.43^{+0.13}_{-0.17}$  (68% CL) obtained by the cosmography [105]. (iii) The density parameters  $\Omega_D$  and  $\Omega_m$  increases and decreases, respectively, as  $z$  decreases. (iv) The effective EoS parameter,  $\omega_{\text{eff}}$ , for the all models, starts from an early matter-dominated regime (i.e.  $\omega_{\text{eff}} = 0$ ) and in the late time,  $z \rightarrow -1$ , it behaves like the  $\Lambda$ CDM model,  $\omega_{\text{eff}} \rightarrow -1$ . (v) The EoS parameter of DE,  $\omega_D$ , for the all models starts at the phase of a cosmological constant, i.e.  $\omega_D = -1$ , and evolves from the phantom phase,  $\omega_D < -1$ , to the non-phantom (quintessence) phase,  $\omega_D > -1$ . The crossing of the phantom divide line  $\omega_D = -1$  occurs in the near past as well as farther future. At late times ( $z \rightarrow -1$ ),  $\omega_D$  approaches again to  $-1$  like the  $\Lambda$ CDM model. Moreover, the present values of  $\omega_D$  for the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB  $f(R)$  models are obtained as  $\omega_{D_0} = -0.94, -0.98, -0.93, -0.94$  and  $-0.97$ , respectively. These values satisfy the present observational constraints [2, 4].

(vi) The variation of the GSL shows that it holds for the aforementioned models from early times to the present epoch. But in the farther future, the GSL for the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB  $f(R)$  models is violated for  $-0.996 < z < -0.955$ ,  $-0.935 < z < -0.909$ ,  $-0.897 < z < -0.751$ ,  $-0.997 < z < -0.958$  and  $-0.995 < z < -0.950$ , respectively. To investigate this problem in ample detail, using Eq. (17) we rewrite Eq. (30) in terms of  $\omega_{\text{eff}}$  as

$$T_A \dot{S}_{\text{tot}} = \frac{1}{4G} \left[ \frac{9}{2} (1 + \omega_{\text{eff}})^2 F + \frac{3}{2} (1 + \omega_{\text{eff}}) \frac{\dot{F}}{H} - (1 + 3\omega_{\text{eff}}) \frac{\ddot{F}}{H^2} \right], \quad (55)$$

which shows that in the farther future  $z \rightarrow -1$  when  $\omega_{\text{eff}} \rightarrow -1$  (see Figs. 1-5), we have

$$T_A \dot{S}_{\text{tot}} \simeq \frac{\ddot{F}}{2GH^2}. \quad (56)$$

According to Eq. (56), the validity of GSL, i.e.  $T_A \dot{S}_{\text{tot}} \geq 0$ , depends on the sign of  $\ddot{F}$ . In Figs. 1-5, we plot the variation of  $\ddot{F}/(2H^2)$  versus  $z$  in the farther future for the selected  $f(R)$  models. Figures confirm that when the sign of  $\ddot{F}$  changes from positive to negative due to the dominance of DE over non-relativistic matter then the GSL is violated. As we know that the natural tendency of any system is to evolve toward thermodynamic equilibrium which is characterized by a state of maximum entropy. In the context of an ever expanding FRW universe, this translates in that the entropy of the apparent horizon plus that of matter and fields enclosed by it must fulfill the GSL of thermodynamics, i.e.  $T_A \dot{S}_{\text{tot}} \geq 0$ . Thus, the violation of the GSL in a  $f(R)$  model means that the model does not approach thermodynamic equilibrium at late times [106]. Of course, as we already mentioned, the GSL can be used as a powerful tool to set bounds on cosmological  $f(R)$  models [86]. It means that we can set the parameters of a given  $f(R)$  model so that the GSL holds throughout the evolution of the universe. Although the parameters used for each model in Figs. 1-5 are the viable ones, by more fine tuning the model parameters the GSL can be held and consequently the model approaches thermodynamic equilibrium at late times. For instance, in AB  $f(R)$  model by choosing the model parameter as  $b = 1.3$ , the GSL is always satisfied from early times to the late cosmological history of the universe.

In Figs. 6-10, we plot the evolutions of  $RF_R$ ,  $G_{\text{eff}}/G$ ,  $g$  and the growth factor  $f$  versus  $z$  for the selected  $f(R)$  models. Figures show that: (i)  $RF_R$  goes to zero for higher values of  $z$  which means that the  $f(R)$  models at high  $z$  regime behave like the  $\Lambda$ CDM model. (ii) The screened mass function  $G_{\text{eff}}/G$  for a given wavenumber  $k$  is larger than one which makes a faster growth of the structures compared to the GR. However, for the higher redshifts, the screened mass function approaches to unity in which the GR structure formation is recovered. Note that the deviation of  $G_{\text{eff}}/G$  from unity for small scale structures (larger  $k$ ) is greater than large scale structures (smaller  $k$ ). (iii) The linear density contrast relative to its value in a pure matter model  $g = \delta/a$  starts from an early matter-dominated phase, i.e.  $g \simeq 1$  and decreases during history of the universe. For a given  $z$ ,  $g$  in the all  $f(R)$  models is greater than that in the  $\Lambda$ CDM model. (iv) The evolution of the growth factor  $f(z)$  for  $f(R)$  models and  $\Lambda$ CDM model together with the 11 observational data of the growth factor listed in Table 1 show that for smaller structures (larger  $k$ ), the all  $f(R)$  models deviate from the observational data. But for larger structures (smaller  $k$ ), the growth factor in the all  $f(R)$  models, very similar to the  $\Lambda$ CDM model, fits the data very well.

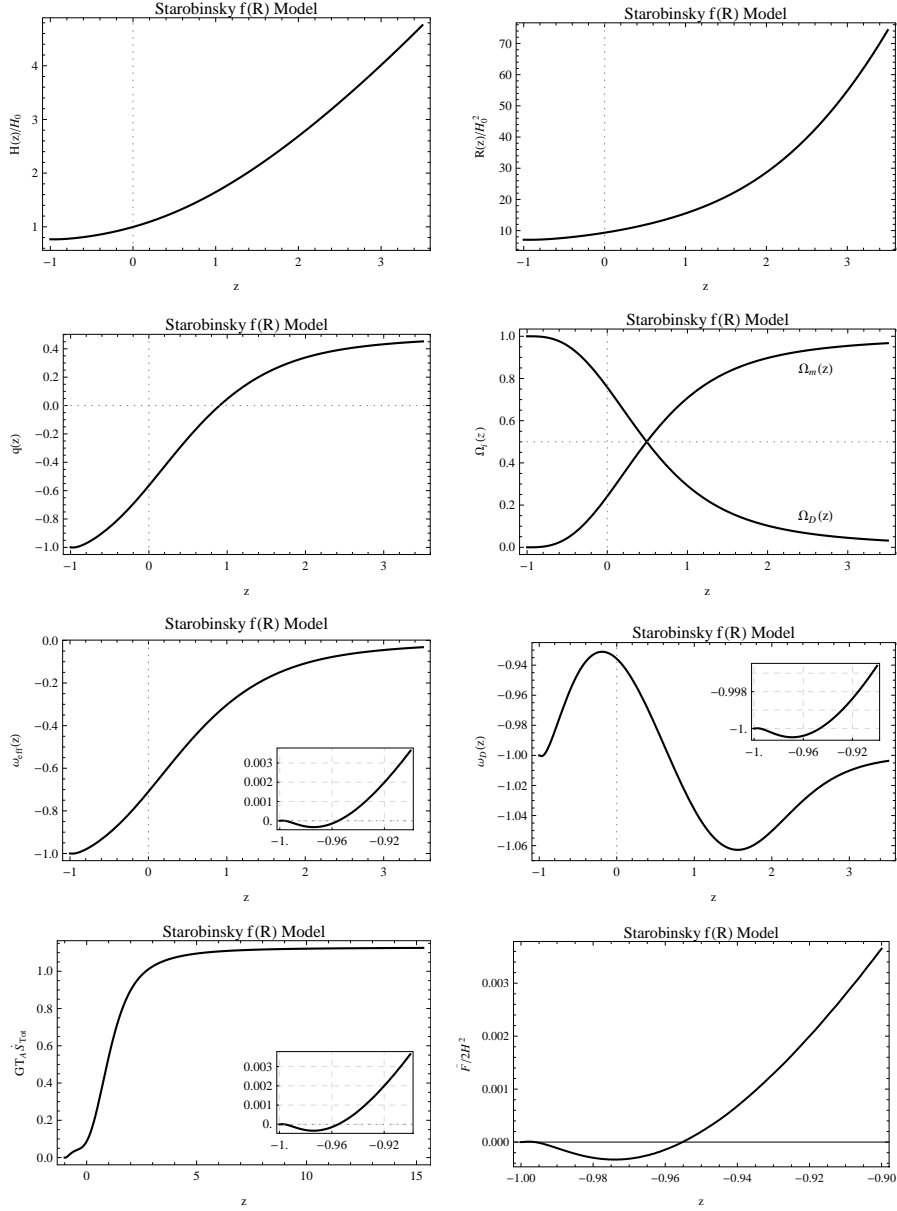


Figure 1: The variations of the Hubble parameter  $H/H_0$ , the Ricci scalar  $R/H_0^2$ , the deceleration parameter  $q$ , the density parameter  $\Omega_i$ , the effective EoS parameter  $\omega_{\text{eff}}$ , the EoS parameter of DE  $\omega_D$ , the GSL,  $GT_A \dot{S}_{\text{tot}}$  and  $\frac{\ddot{F}}{2H^2}$  versus redshift  $z$  for the Starobinsky model. Auxiliary parameters are  $\Omega_{m_0} = 0.24$ ,  $\Omega_{D_0} = 0.76$ ,  $\Omega_{\text{rad}_0} = 4.1 \times 10^{-5}$ ,  $\lambda = 1$  and  $n = 2$ .

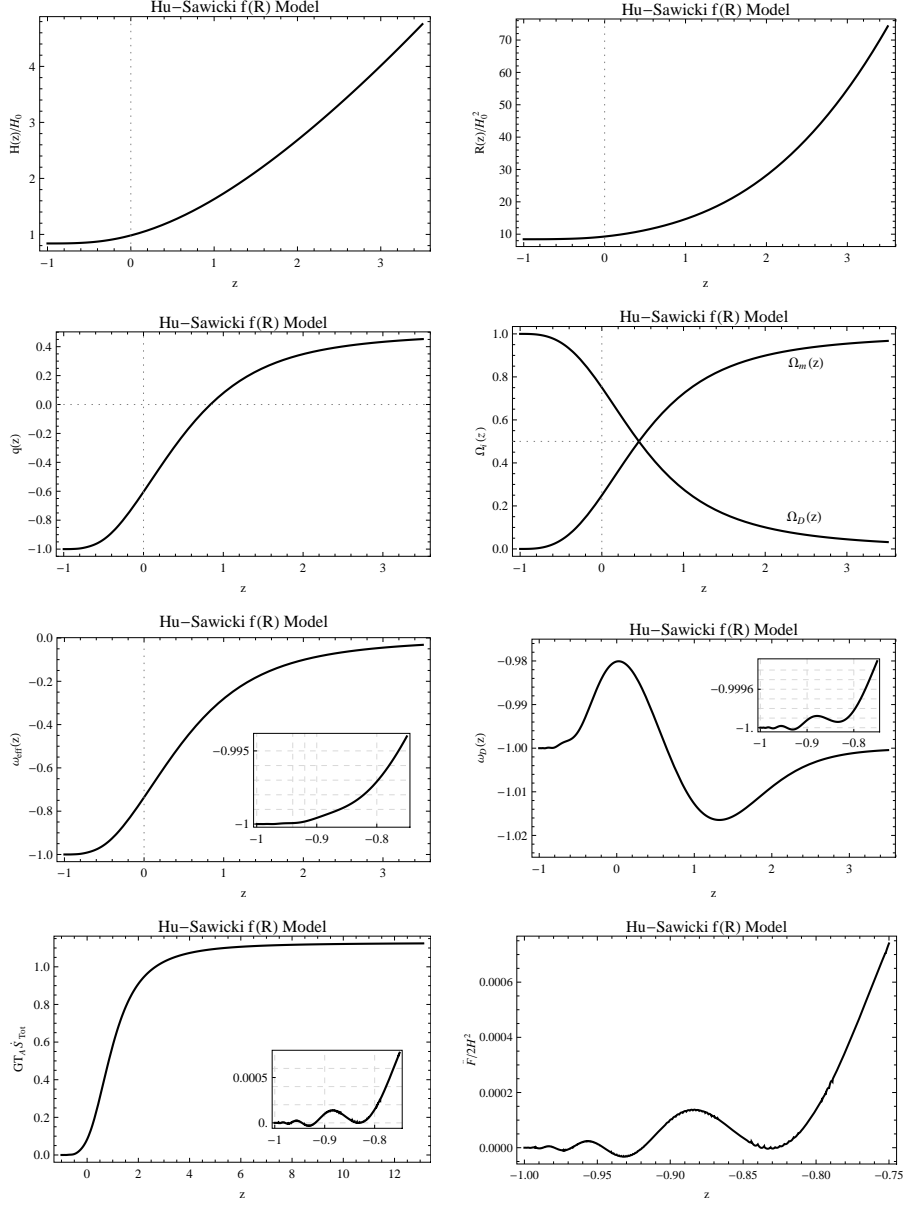


Figure 2: Same as Fig. 1 but for the Hu-Sawicki model. Auxiliary parameters are  $\Omega_{m_0} = 0.24$ ,  $\Omega_{D_0} = 0.76$ ,  $\Omega_{\text{rad}_0} = 4.1 \times 10^{-5}$ ,  $c_1 = 1.25 \times 10^{-3}$ ,  $c_2 = 6.56 \times 10^{-5}$  and  $n = 4$ .

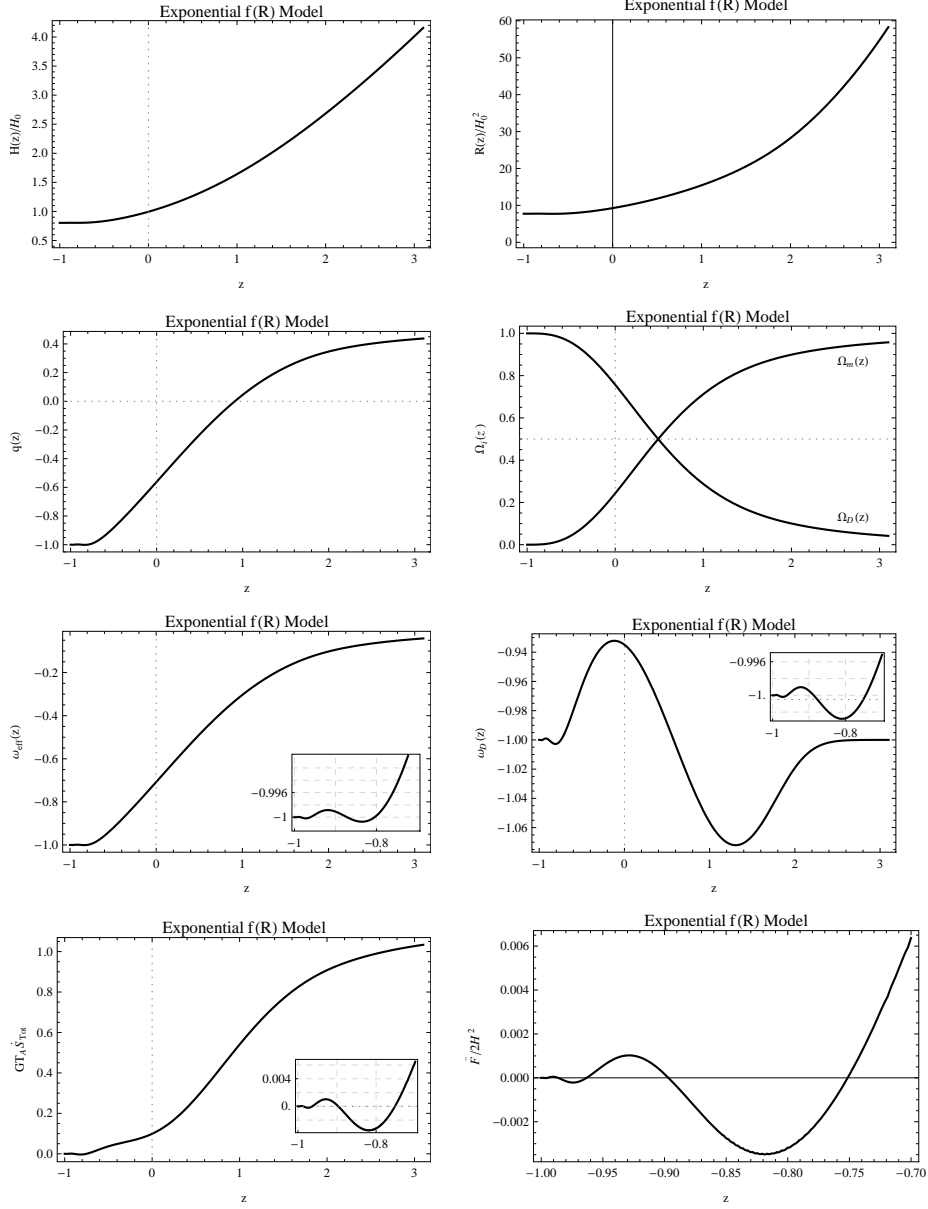


Figure 3: Same as Fig. 1 but for the Exponential model. Auxiliary parameters are  $\Omega_{m_0} = 0.24$ ,  $\Omega_{D_0} = 0.76$ ,  $\Omega_{\text{rad}_0} = 4.1 \times 10^{-5}$  and  $\beta = 1.8$ .

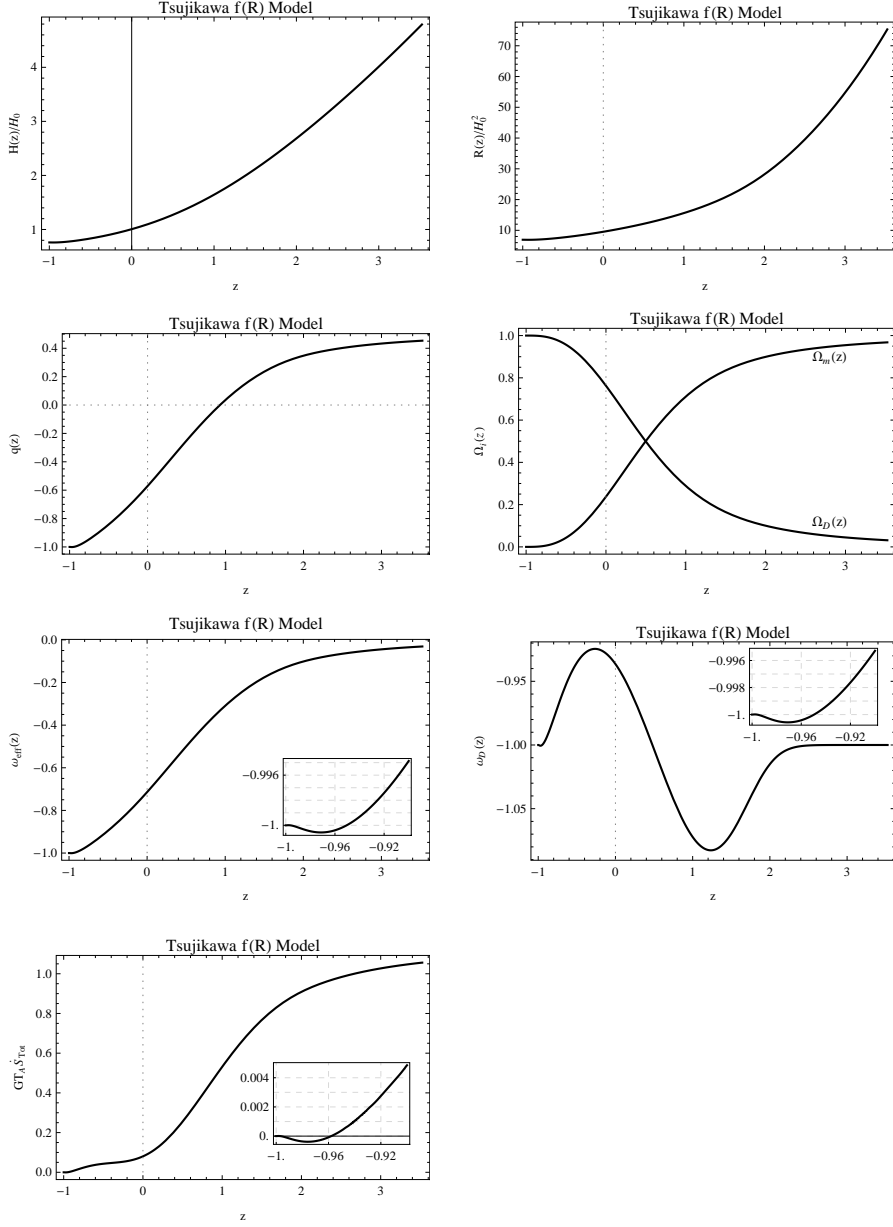


Figure 4: Same as Fig. 1 but for the Tsujikawa model. Auxiliary parameters are  $\Omega_{m_0} = 0.24$ ,  $\Omega_{D_0} = 0.76$ ,  $\Omega_{\text{rad}_0} = 4.1 \times 10^{-5}$  and  $\lambda = 1$ .

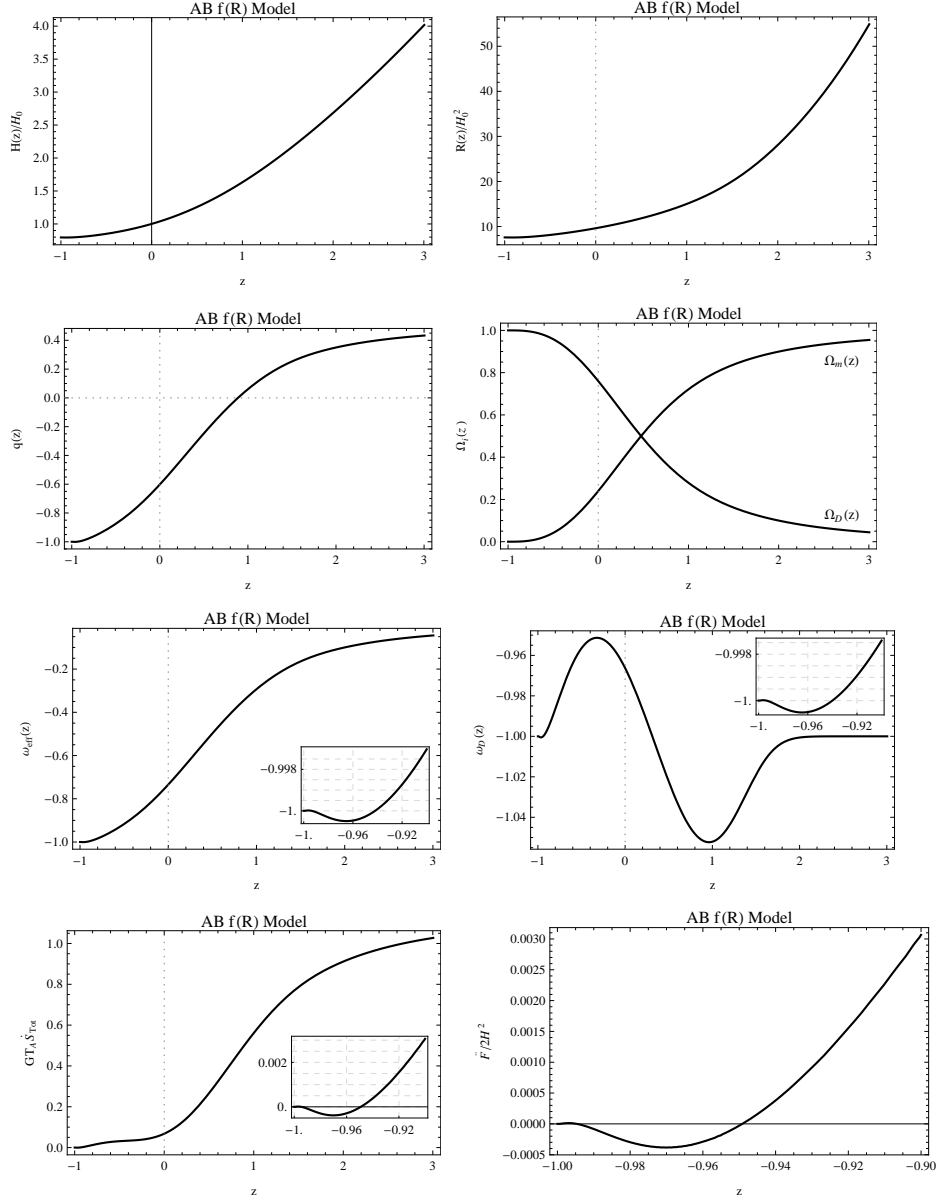


Figure 5: Same as Fig. 1 but for the AB model. Auxiliary parameters are  $\Omega_{m_0} = 0.24$ ,  $\Omega_{D_0} = 0.76$ ,  $\Omega_{\text{rad}_0} = 4.1 \times 10^{-5}$  and  $b = 1.4$ .



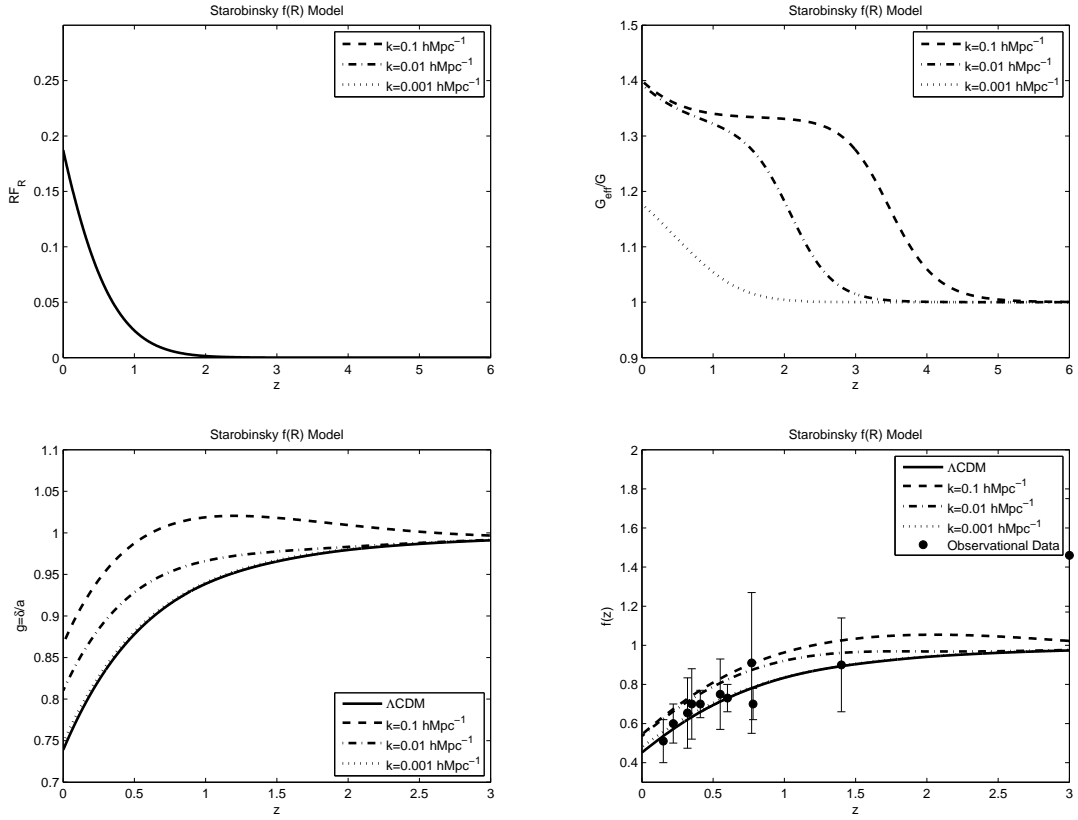


Figure 6: The variations of  $RF_R$ , the screened mass function  $G_{\text{eff}}/G$ , the linear density contrast relative to its value in a pure matter model  $g = \delta/a$  and the growth factor  $f(z)$ , versus redshift  $z$  for the Starobinsky model.

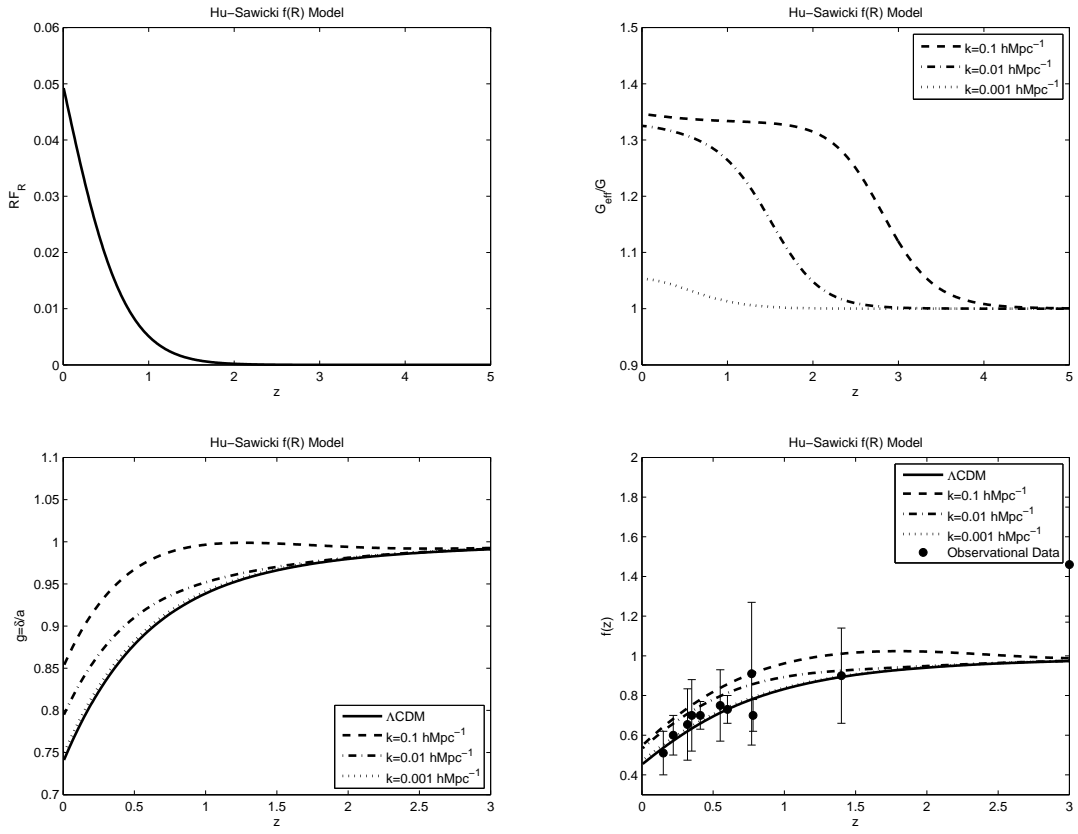


Figure 7: Same as Fig. 6 but for the Hu-Sawicki model.

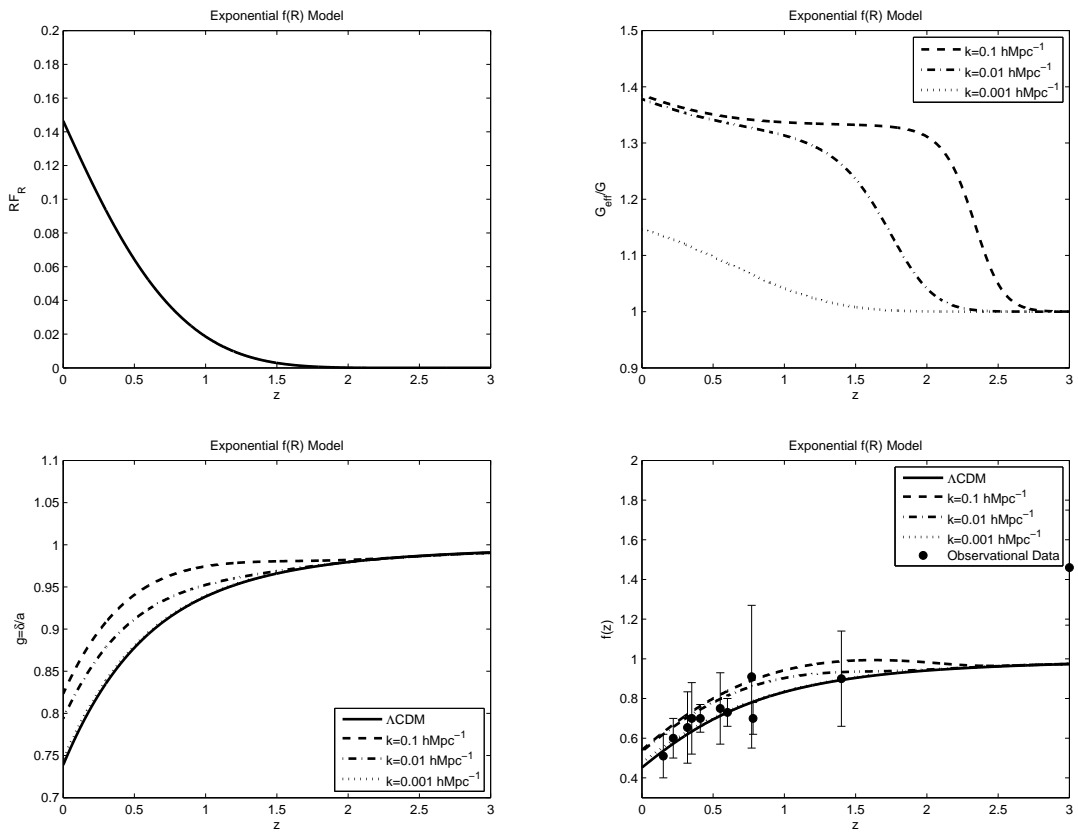


Figure 8: Same as Fig. 6 but for the Exponential model.

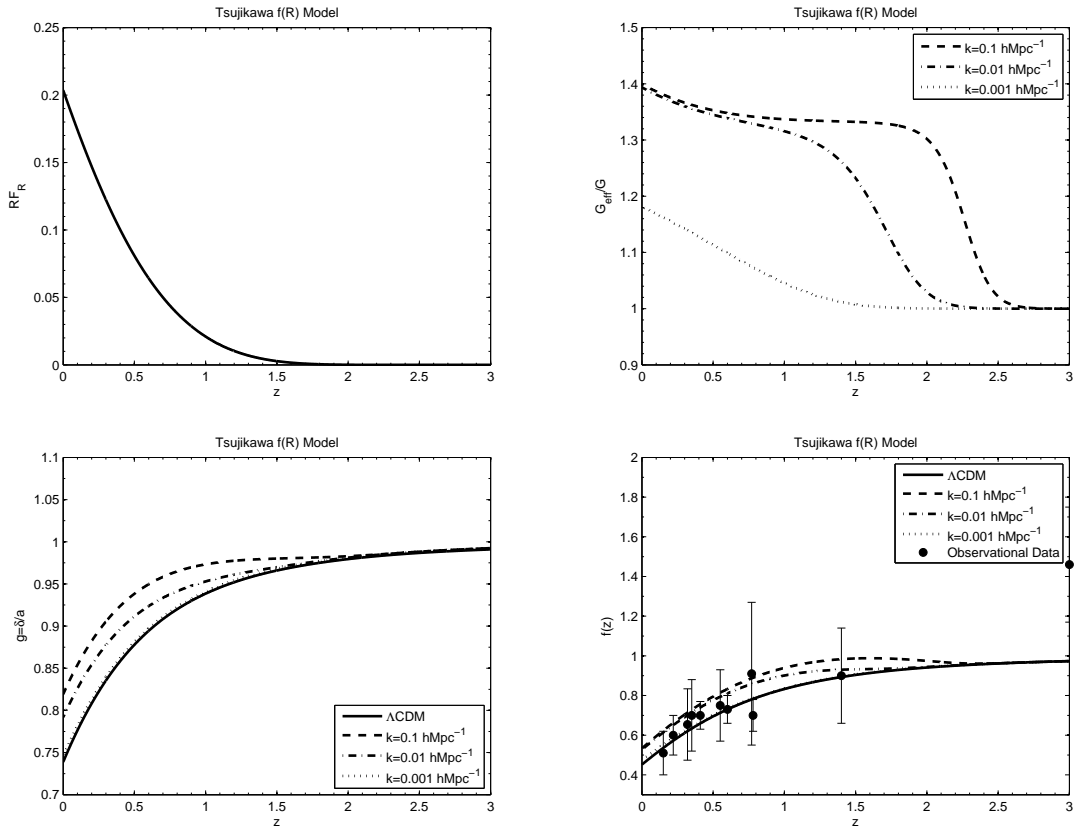


Figure 9: Same as Fig. 6 but for the Tsujikawa model.

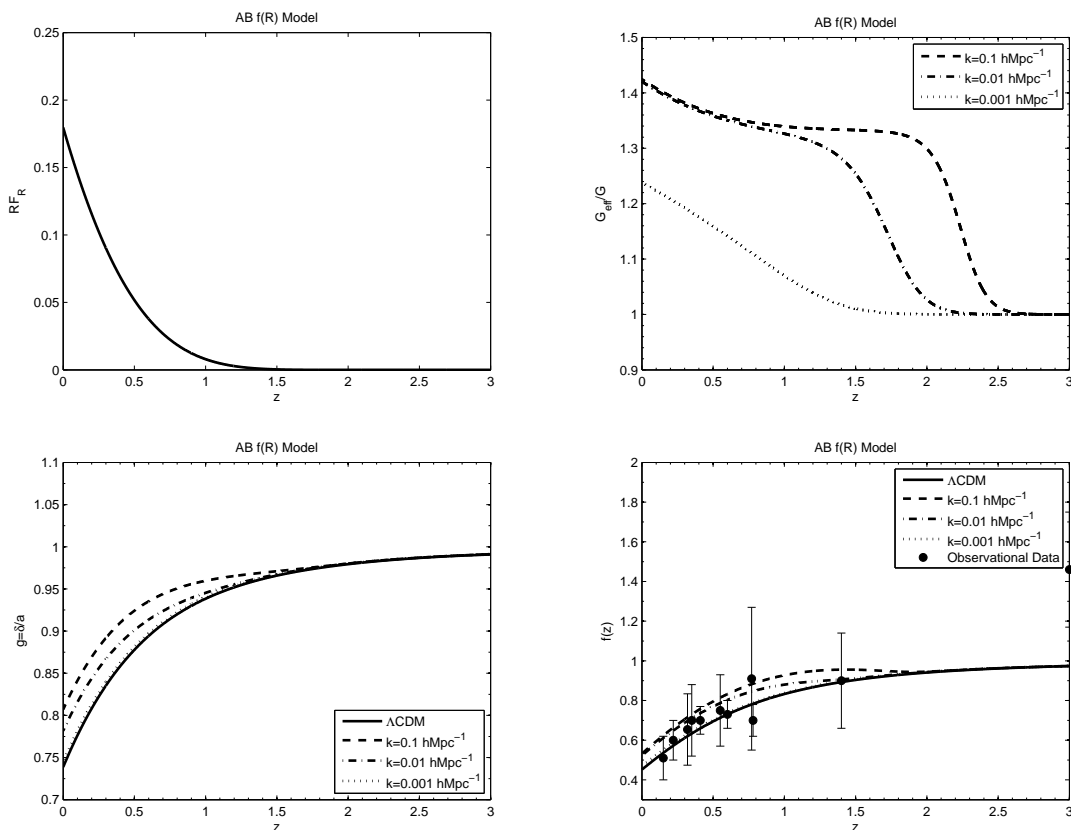


Figure 10: Same as Fig. 6 but for the AB model.

Table 1: The observational data for the linear growth rate  $f_{\text{obs}}(z)$ .

$z$	0.15	0.22	0.32	0.35	0.41	0.55	0.60	0.77	0.78	1.4	3.0
$f_{\text{obs}}$	0.51	0.60	0.654	0.70	0.70	0.75	0.73	0.91	0.70	0.90	1.46
$1\sigma$	0.11	0.10	0.18	0.18	0.07	0.18	0.07	0.36	0.08	0.24	0.29
Ref.	[107]	[110]	[111]	[112]	[110]	[113]	[110]	[114]	[110]	[115]	[116]

## 8 Conclusions

Here, we investigated the evolution of both matter density fluctuations and GSL in some viable  $f(R)$  models containing the Starobinsky, Hu-Sawicki, Exponential, Tsujikawa and AB models. For the aforementioned models, we first obtained the evolutionary behaviors of the Hubble parameter, the Ricci scalar, the deceleration parameter, the matter and DE density parameters, the EoS parameters and the GSL. Then, we explored the growth of structure formation in the selected  $f(R)$  models. Our results show the following.

(i) All of the selected  $f(R)$  models can give rise to a late time accelerated expansion phase of the universe. The deceleration parameter for all models shows a cosmic deceleration  $q > 0$  to acceleration  $q < 0$  transition. The present value of the deceleration parameter takes place

in the observational range. Also, at late times ( $z \rightarrow -1$ ), it approaches a de Sitter regime (i.e.  $q \rightarrow -1$ ), as expected.

(ii) The effective EoS parameter  $\omega_{\text{eff}}$  for the all models starts from the matter dominated era,  $\omega_{\text{eff}} \simeq 0$ , and in the late time,  $z \rightarrow -1$ , it behaves like the  $\Lambda$ CDM model,  $\omega_{\text{eff}} \rightarrow -1$ .

(iii) The evolution of the EoS parameter of DE,  $\omega_D$ , shows that the crossing of the phantom divide line  $\omega_D = -1$  appears in the near past as well as farther future. This is a common physical phenomena to the existing viable  $f(R)$  models and thus it is one of the peculiar properties of  $f(R)$  gravity models characterizing the deviation from the  $\Lambda$ CDM model [101].

(iv) The GSL is respected from the early times to the present epoch. But in the farther future, the GSL for the all models is violated in some ranges of redshift. The physical reason why the GSL does not hold in the farther future is that the sign of  $\ddot{F}$  changes from positive to negative due to the dominance of DE over non-relativistic matter.

(v) For all models, the screened mass function  $G_{\text{eff}}/G$  is larger than 1 and in high  $z$  regime goes to 1. The deviation of  $G_{\text{eff}}/G$  from unity for larger  $k$  (smaller structures) is greater than the smaller  $k$  (larger structures). The modification of GR in the framework of  $f(R)$ -gravity gives rise to an effective gravitational constant,  $G_{\text{eff}}$ , which is time and scale dependent parameter in contrast to the Newtonian gravitational constant.

(vi) The linear density contrast relative to its value in a pure matter model,  $g(a) = \delta_m/a$ , for all models starts from an early matter-dominated phase,  $g(a) = 1$ , and decreases during history of the universe.

(vii) The evolutionary behavior of the growth factor of linear matter density perturbations,  $f(z)$ , shows that for all models, the growth factor for smaller  $k$  (larger structures) like the  $\Lambda$ CDM model fits the data very well.

It is worth noting that the  $f(R)$ -gravity for very small wavenumbers (larger structures) is completely indistinguishable from  $\Lambda$ CDM. The main effect of the  $f(R)$  theory is in quasi-linear regimes, large wavenumbers (smaller structures) where the growth rate has a strong scale dependence and deviates from the standard  $\Lambda$ CDM case. Also, for any given wavenumber corresponding to the larger/smaller structures, the  $f(R)$  model can have a growth function identical to  $\Lambda$ 's at high redshift. Future surveys of the large scale structure such as eBOSS, DESI, Euclid, or WFIRST [26] may reveal the growth index in terms of wavenumber of the structures and help the  $f(R)$ -gravity models to be clearly distinguished from the  $\Lambda$ CDM model.

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