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# Super Algebra and Harmonic Oscillator in Anti de Sitter Space

Shahin Toni[1] · Omid Jalili[2]

[1] Department of Physics, Islamic Azad University-Nour Branch, P.O.Box 4641738373, Nour, Mazandaran, IRAN;
email: shahin.toni75@gmail.com

[2] Department of Physics, Islamic Azad University-Nour Branch, P.O.Box 4641738373, Nour, Mazandaran, IRAN;
email: omid_jalili@yahoo.com

**Abstract**. The harmonic oscillator in anti de Sitter space(AdS) is discussed. We consider the harmonic oscillator potential and then time independent Schrodinger equation in AdS space. Then, we apply the supersymmetric Quantum Mechanics approach to solve our differential equation. In this paper, we have solved Schrodinger equation for harmonic oscillator in AdS spacetime by supersymmetry approach. The shape invariance, charge conjugation and other properties of supersymmetric Quantum Mechanics for our equation are discussed. We investigate the dynamical symmetry into definite group. We also obtain the Casimir operator and thermodynamics properties of harmonic oscillator in Anti de sitter space. We have discussed the corresponding algebra for the $N = 2$ Supersymmetry. The energy spectrum of the harmonic oscillator is similar to the flat space but the constant terms of it are different, This constant terms depend on the geometrical parameter of the background. Finally, we have obtained the various thermodynamics quantity of the system.

*Keywords*: Harmonic Oscillator, Anti de Sitter space, Supersymmetry, Thermal Properties

# 1 Introduction

Anti de Sitter space(AdS) is a curved space time with a constant negative curvature. It is solution of Einstein equation with cosmological constant. On the other hand, the degeneracy in the space of constant curvature is discussed for the first time by Schrodinger [1, 2], Infeld [3] and Stivenson [4]. This subject has interested many researcher. Because it connects to nontrivial realization of hidden symmetry and also is applied to construct the many particle Wave functions [5], non relativistic models of quark system [6, 7] and solutions of two-center problem [8]. They have shown that the complete degeneracy of spectrum of the coulomb problem and harmonic oscillator on the three dimensional sphere in the orbital and azimuthal quantum number is caused by an additional integral of motion. However, in contrast with the flat space, the integrals of motion for the coulomb problem and isotropic oscillator do not have simple algebra as the relevant commutators. The group of hidden symmetry of this systems with accidental degeneracy give us motivation to discuss the isotropic oscillator with constant positive curvature. Also note that the concept of shape invariance has extended to ordinary differential equations. In that case, the second order differential operator will decompose the multiplication of raising and lowering operators [9, 16]. In this paper, we use the factorization method and shape invariance of the associated Jacobi equation with respect to parameters n and m and obtain the factorized schrodinger equations for the harmonic oscillator in constant negative curvature. Also, we obtain the shape invariance relation for the corresponding potential. We should stress that supersymmetry(SUSY) in quantum mechanic(QM) by factorization method is based upon the framework of shape invariance. If one quantum mechanic problem obtains supersymmetry tentative framework, we can then factorize hamiltonian in terms of a multiplication of first order differential operator. Also, note that the shape invariance condition must be satisfied. In this approach, hamiltonian is the product of raising and lowering operators. The organization of this paper is as follows. In section 2, we discuss AdS in an appropriate coordinate. In section 3, we obtain the energy spectrum of the harmonic oscillator in AdS space. The generalized SUSY is discussed in section 4. We obtain super charge and other supersymmetric quantities in section 5. The thermodynamics of the system is discussed in section 6. We end with a discussion.

## 2 Anti de Sitter space

As we know, the three-dimensional space of constant negative curvature can also be realized geometrically on the three-dimensional sphere S of the radius R embedded in to the four-dimensional pseudo-Euclidean space;

$$-(x^0)^2 - (x^3)^2 + (x^2)^2 + (x^1)^2 = -R^2. \tag{1}$$

Actually, the metric of this ambient space is flat :

$$ds^2 = -(dx^0)^2 - (dx^3)^2 + (dx^2)^2 + (dx^1)^2 \tag{2}$$

We can work in intrinsic space by the following coordinate transformations,

$$x^0 = R\cosh\rho\cos\tau, \quad x^3 = R\cosh\rho\ sin\tau, \quad x^1 = R\sinh\rho\Omega_1, \quad x^2 = R\sinh\rho\Omega_2 \tag{3}$$

where $0 \leq \Omega \leq 2\pi$, $0 \leq \rho$ and $\Omega_1^2 + \Omega_2^2 = 1$. Then, the metric in the intrinsic space will be,

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_i). \tag{4}$$

and in matrix form,

$$g_{\mu\nu} = R^2 \begin{pmatrix} -cosh^2\rho & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & sinh^2\rho & 0 \\ 0 & 0 & 0 & sinh^2\rho \end{pmatrix}. \tag{5}$$

Now, we are going to write the general form of isotropic oscillator potential in AdS.

## 3 Isotropic Oscillator on Anti de Sitter space

The isotropic oscillator potential is:

$$v(R) = \frac{\mu\omega^2 R^2}{2} \tag{6}$$

If we use the $R^2 = x^2 + y^2$ for distance(we works here in two space like dimension), we can find potential bellow in the intrinsic space [16],

$$v(R) = \frac{\mu\omega^2 R^2 \tanh^2\rho}{2} \tag{7}$$

Using this potential, we then need to obtain the Laplace Beltrami operator to apply the results in the Schrodinger equation $\Delta\Psi + V\Psi = E\Psi$. The Laplace Beltrami operator is

$$\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\sqrt{|g|}g^{ik}\frac{\partial}{\partial x^k}. \tag{8}$$

where $g = \det(g_{ij})$. In our intrinsic space, the $g$ is,

$$g = -R^6\cosh^2\rho\sinh^4\rho \tag{9}$$

and

$$\sqrt{|g|} = R^3\cosh\rho\sinh^2\rho. \tag{10}$$

By using eq. (4)(5)(8)(9)(10) laplace operator gives

$$\Delta\psi = -R^2\cosh^2\rho\frac{\partial^2\psi}{\partial^2\tau} + R^2\frac{\partial^2\psi}{\partial^2\rho} + R^2\tanh\rho\frac{\partial\psi}{\partial\rho} + 2R^2\cosh\rho\frac{\partial\psi}{\partial\rho} - R^2\sinh^2\rho\frac{\partial^2\psi}{\partial^2\Omega_i} \tag{11}$$

Thus, one can rewrite the Schrodinger equation as,

$$-\cos^2 h\rho\frac{\partial^2\psi}{\partial^2\tau}+\frac{\partial^2\psi}{\partial^2\rho}+\tanh\rho\frac{\partial\psi}{\partial\rho}+2\coth\rho\frac{\partial\psi}{\partial\rho}-R^2\sin^2 h\rho\frac{\partial^2\psi}{\partial^2\Omega_i}+\frac{2\mu}{\hbar^2}(E-\frac{1}{2}\mu\omega^2R^2\tan^2 h\rho)\psi = 0 \tag{12}$$

Now, we decompose the wave function as $\psi(\tau,\rho,\Omega) = R(\tau)F(\Omega)\Phi(\rho)$, then we have

$$\frac{-\cos^2 h\rho}{R(\tau)}\frac{d^2\rho}{d^2\tau} + \frac{1}{\Phi(\rho)}\frac{d^2\Phi(\rho)}{d^2\rho} + \tanh\rho\frac{1}{\Phi(\rho)}\frac{d\Phi(\rho)}{d\rho} + 2\coth\rho\frac{1}{\Phi(\rho)}\frac{d\Phi(\rho)}{d\rho}$$
$$-R^2\sin^2 h\rho\frac{1}{F(\Omega_i)}\frac{d^2F}{d^2\Omega_i} + \frac{2\mu}{\hbar^2}(E - \frac{1}{2}\mu\omega^2R^2\tan^2 h\rho) = 0 \tag{13}$$

First of all we take

$$\frac{1}{R(\tau)}\frac{d^2R}{d\tau^2} = \eta^2, \quad R(\tau) = ce^{\eta\tau} \tag{14}$$

Also , in second step we take

$$\frac{1}{F}\frac{d^2F}{d\Omega_i^2} = -M^2, \quad F(\Omega_i) = De^{iM\omega_i} \tag{15}$$

So, finally the corresponding schrodinger equation in case of $M^2 = \mu^2$ will be as,

$$\frac{d^2\Phi(\rho)}{d\rho^2} + (\tanh\rho + 2\coth\rho)\frac{d\Phi}{d\rho} + \frac{2\mu}{\hbar^2}(E + \frac{\hbar^2M^2}{2\mu} - \frac{1}{2}\mu\omega^2R^2\tanh^2\rho)\Phi(\rho) = 0 \tag{16}$$

In order to solve this equation and obtain the energy and also the wave function, one can choose the following of variable,

$$\tanh\rho = x \tag{17}$$

Using this change of variable, we arrive at following equation,

$$(1-x^2)\Phi''(x)+(\frac{2}{x}-x)\Phi'(x)+\frac{2\mu}{\hbar^2}\frac{(E+\frac{\hbar^2M^2}{2\mu}-\frac{1}{2}\mu\omega^2R^2x^2)}{1-x^2}\Phi(x)=0 \qquad (18)$$

Now, we are going to employ the factorization method to simplicity solve the above equation [20]. In that case, we take $\Phi(x)=P^{(\alpha,\beta)}_{(n,m)}(x)G(x)$ so that the equation (18) can be written as following,

$$(1-x^2)P''^{(\alpha,\beta)}_{(n,m)}(x)+[(1-x^2)2\frac{G'}{G}+(\frac{2}{x}-x)]P'^{(\alpha,\beta)}_{(n,m)}(x)$$
$$+[(1-x^2)\frac{G''}{G}+(\frac{2}{x}-x)\frac{G'}{G}+\frac{2\mu}{\hbar^2}\frac{(E+\frac{\hbar^2M^2}{2\mu}-\frac{1}{2}\mu\omega^2R^2x^2)}{1-x^2}]P^{(\alpha,\beta)}_{(n,m)}(x)=0 \qquad (19)$$

Here, we call the associated Jacobi equation, which is given by [15],

$$(1-x^2)P''^{(\alpha,\beta)}_{(n,m)}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P'^{(\alpha,\beta)}_{(n,m)}(x)$$
$$+[\frac{n(\alpha+\beta n+1)-n(\alpha+\beta+n+1)x^2-m[\alpha+\beta+m+(\alpha-\beta)]x}{1-x^2}]P^{(\alpha,\beta)}_{(n,m)}(x)=0. \qquad (20)$$

Now, we compare last two equations terms by terms. By comparing we deduce that: $\frac{G'}{G}=\frac{1}{x(1-x^2)}+\frac{\beta-\alpha}{2(1-x^2)}-\frac{(\alpha+\beta+1)x}{2(1-x^2)}$. By solving this Eq, we find $G$ function as $G(x)=G(0)[x(1+x)^{\frac{2\beta-1}{4}}(1-x)^{\frac{2\alpha-1}{4}}]$ So, in that case the wave function will be as,

$$\Phi(x)=G(0)[x(1+x)^{\frac{2\beta-1}{4}}(1-x)^{\frac{2\alpha-1}{4}}]P^{(\alpha,\beta)}_{(n,m)}(x) \qquad (21)$$

By comparing the third term of Eq.(19) and Eq.(20), one can obtain the followings result. Here, we obtain the condition as $\alpha+\beta+1=\frac{\mu^2\omega^2R^2}{\hbar^2}$ and $\alpha=\beta$ and $\alpha=-\beta$. Here, we can not use $\alpha=-\beta$, because for our special function such condition does not make sense. So, the above condition lead us to arrange the energy spectrum as,

$$E_n=\frac{\hbar^2}{2\mu}[n(n+\frac{\mu^2\omega^2R^2}{\hbar^2})+3\frac{\mu^2\omega^2R^2}{\hbar^2}+1-\frac{\hbar^2M^2}{2\mu}] \qquad (22)$$

So, generally here we could to arrange the energy spectrum and wave function.

## 4 SUSY quantum mechanics

Next step, we have to discuss the supersymmetry approaches with factorization method. We take advantage from [10] , and obtain the $A_{n,m}, B_{n,m}$ operators which is given by,

$$B_{(n,m)}A_{(n,m)}\Phi_{n,m}(x)=N_{(n,m)}\Phi_{n,m}(x) \qquad A_{(n,m)}B_{(n,m)}\Phi_{n-1,m}(x)=N_{(n,m)}\Phi_{n-1,m}(x) \qquad (23)$$

From equation Eq. (23), one can obtain directly following recursion relations,

$$B_{(n,m)}\Phi_{n-1,m}(x)=\Phi_{n,m}(x)$$
$$A_{(n,m)}\Phi_{n,m}(x)=N_{(n,m)}\Phi_{n-1,m}(x) \qquad (24)$$

We use the above assumption, then the raising and lowering operators will be as,

$$A_{(n,m)}(x) = (1-x^2)\frac{d}{dx} + nx \tag{25}$$

$$B_{(n,m)}(x) = -(1-x^2)\frac{d}{dx} + (n+2\alpha)x \tag{26}$$

# 5 Dynamical symmetry into definite group

Next, we are going to write the supercharge as,

$$Q = \begin{pmatrix} 0 & 0 \\ A_{n,m} & 0 \end{pmatrix} \tag{27}$$

$$Q^{\dagger} = \begin{pmatrix} 0 & B_{n,m} \\ 0 & 0 \end{pmatrix} \tag{28}$$

So, the bosonic and fermionic hamiltonians $H_1$ and $H_2$ will be as,

$$\begin{aligned} H_1 &= A_{n,m}(\rho)B_{n,m}(\rho) \\ H_2 &= B_{n,m}(\rho)A_{n,m}(\rho) \end{aligned} \tag{29}$$

As we know, the element of space part of Eq.(4) will be as,

$$dL^2 = R^2(d\rho^2 + \sinh^2\rho d\Omega^2) \tag{30}$$

During our calculation, we have seen the master function $A(x) = 1 - x^2$ and also the weight function $W(x) = (1+x)^{\frac{2\beta-1}{4}}(1-x)^{\frac{2\alpha-1}{4}}$ with $\alpha > -\frac{\beta}{2}$ and $\beta > -\frac{3}{2}$, in the the new coordinate as we mentioned, $x = \tanh\rho$. So, in order to investigate the dynamical symmetry group, we have consider the Eq. (14) of the [19]. Here, note that we have chosen an appropriate coordinate, we can assume that $A''(x) = -2$ and the non- vanishing Ricci tensors are $R_{\rho\rho} = 1$ ,$R_{\Omega\Omega} = \sinh^2\rho$. Thus, the Ricci curvature is $R = \frac{2}{a^2}$ By using this information and Eq.(14) of [19], one can obtain the dynamical symmetry group with the following operators,

$$L_+ = e^{i\Omega}(\frac{\partial}{\partial\rho} + i\coth\rho\frac{\partial}{\partial\Omega}) \tag{31}$$

$$L_- = e^{-i\Omega}(-\frac{\partial}{\partial\rho} + i\coth\rho\frac{\partial}{\partial\Omega} - \frac{\alpha+\beta}{\tanh\rho}) \tag{32}$$

$$L_3 = -i\frac{\partial}{\partial\Omega}, \quad I = 1. \tag{33}$$

These are generators of SU(2) algebra and they are satisfied by follows commutation relation.

$$[L_+, L_-] = 2L_3 + (\alpha+\beta)I \tag{34}$$

$$[L_3, L_\pm] = \pm L_\pm \tag{35}$$

$$[L, I] = 0 \tag{36}$$

The first relation show us, we have generalized SU(2) algebra. Also, here the Casimir operator will be as [19].

$$C=-\frac{1}{2R^2}[\frac{\partial^2}{\partial\rho^2}+(\frac{\alpha+\beta-1}{\tanh\rho})\frac{\partial}{\partial\rho}+\frac{1}{\sinh^2\rho}\frac{\partial^2}{\partial\Omega^2}-\frac{1}{2}(\alpha+\beta)] \tag{37}$$

The Casimir operator will be generators of equation Eq.(33).

# 6 The thermal properties of system

Now, we are going to obtain the various thermodynamics properties of the system. In order to do such calculation, we need to introduce the partition function of the system. The corresponding energy from equation Eq.(22) leads us to the following partition function,

$$Q=\sum_r e^{-\beta E_r}=\sum_{n=0} e^{-\beta E_n} \tag{38}$$

where for simplicity to manipulate it, we rewrite the energy, Eq(22), as $E_n=\frac{\hbar^2}{2\mu}[n(n+a)+b]$ where $a=\frac{\mu^2\omega^2R^2}{\hbar^2}$ and $b=3\frac{\mu^2\omega^2R^2}{\hbar^2}+1-\frac{\hbar^2M^2}{2\mu}$. Then we change the power as;

$$E_n=\frac{\hbar^2}{2\mu}[n(n+a)+b]=\frac{\hbar^2}{2\mu}[(n+\frac{a}{2})^2-\frac{a^2}{4}+b]. \tag{39}$$

So, we have

$$Q=e^{-\beta\frac{\hbar^2}{2\mu}[-\frac{a^2}{4}+b]}\sum_{n=0} e^{-\beta\frac{\hbar^2}{2\mu}(n+\frac{a}{2})^2}. \tag{40}$$

Neglecting the error of first few value of $n$, we can replace $\sum$ with integral, the end formula for partition function will be

$$Q=e^{-\beta\frac{\hbar^2}{2\mu}[-\frac{a^2}{4}+b]}\sqrt{\frac{\pi\mu}{2\beta\hbar^2}} \tag{41}$$

Now, we can easily derive other thermodynamics quantity. The internal energy will be

$$U=-\frac{\partial}{\partial\beta}\ln Q=\frac{\hbar^2}{2\mu}[-\frac{a^2}{4}+b]+\frac{1}{2\beta}. \tag{42}$$

Here, we see that if we take $\frac{a^2}{4}=b$, one can obtain the internal energy as $U=\frac{1}{2}kT$. As we can see, the internal energy increase with temperature and have a nonzero energy at temperature zero. Helmholtz free energy of the system respectively is as

$$A=-\frac{1}{\beta}\ln Q=-\frac{\hbar^2}{2\mu}[-\frac{a^2}{4}+b]-\frac{1}{2\beta}\ln\frac{\pi\mu}{2\beta\hbar^2} \tag{43}$$

Also, in case of $\frac{a^2}{4}=b$ the corresponding free energy $A$ will be as $A=-\frac{KT}{2}\ln\frac{\alpha}{2}KT$, Where $\alpha$ is $\alpha=\frac{\pi\mu}{\hbar^2}$. The entropy of the system will be obtained by following expression

$$S=-\frac{\partial}{\partial T}A=-\frac{\partial A}{\partial\beta}\frac{\partial\beta}{\partial T}=\frac{1}{KT^2}\frac{\partial A}{\partial\beta}=\frac{K}{2}(\ln\frac{\pi\mu}{2\beta\hbar^2}-1). \tag{44}$$

So, the corresponding entropy disagree with $S=\frac{K}{2}\ln\frac{\alpha}{2}T-S_0$, where $S_0=\frac{K}{2}$.

In that case, one can say that the entropy will increase with temperature as in flat space.

# 7 Conclusions

In this paper, we have solve Schrodinger equation for harmonic oscillator in AdS spacetime by supersymmetry approach. we have discussed the corresponding algebra for the $N = 2$ Supersymmetry. The energy spectrum of harmonic oscillator is similar to flat space but the constant terms of it are different. This constant terms depend on the geometrical parameter of the background. Finally, we have obtained the various thermodynamics quantity of the system.

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